Sheaf Cohomology and Serre Duality

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Abstract

The characterisation problem is central to Algebraic geometry. It aims at characterising all varieties up to isomorphism. Thus, any object that happens to be an invariant of a scheme/sheaf can be exploited to further our understanding of the problem. As such, the cohomology group of any sheaf of modules on a scheme is an invariant of that sheaf. Therefore if two spaces have different cohomology groups we can immediately conclude that they are not isomorphic in the desired category, but one must be cautious about stating the converse. Two non-isomorphic spaces may have the same cohomology groups.

The Serre duality theorem for projective schemes relates the cohomology group and the another functor known as the Ext functor. It states that given any $\mathsf{coherent\ sheaf}$ on a Cohen-Macaulay scheme, the i^{th} Ext functor of that sheaf to the dualizing sheaf (denoted by ω - which is a coherent sheaf unique to a proper scheme) is isomorphic to the dual of the $n - i^{th}$ cohomology groups of the sheaf. The duality theorem plays an important role in proving other theorems like the Riemann-Roch theorem for curves and giving an equivalent statement for the Kodaira Vanishing theorem (which shall be discussed in the thesis).

The Kodaira Vanishing Theorem says that for any projective non-singular variety X of dimension n over $\mathbb{C}$ and any ample invertible sheaf $\mathcal{I}$, the i^{th} cohomology group of $\mathcal{I} \bigotimes \omega$ is zero for all $i > 0$. We shall no be seeing the proof of the Kodaira Vanishing Theorem, but instead shall see how the Serre duality theorem gives rise to an equivalent statement which says that the k^{th} cohomology group of $\mathcal{I}^{-1}$ is zero for all $k < n$

Acknowledgement

Firstly, I would like to thank the person who put in the most effort in seeing to it that I understood the topic at hand and sat patiently clearing every doubt I had and helped guide me throughout this project, Prof. Dr. Krishna Hanumanthu. I strongly believe that I would have made little to no progress without his guidance. I would also like to take this opportunity to thank all the faculty at Shiv Nadar University (SNU) who put in a lot of effort into our courses and also helped motivate us. I would specifically like to thank Dr. Debashish Bose, Dr. Pradip Kumar and Dr. Satyanarayana Reddy who always made themselves available to discuss and clear any doubt I had.

Next, I wish to thank all my friends at SNU, who created a huge impact in my academic and personal life. Although it would be unwise to take names, I believe some people do deserve special mentions. I would like to thank Ujan Banerjee for being a great roommate and a wonderful friend. All the silly things we have a done and said will always be memories I cherish. I would also like to thank Harsh Prasad, Nishank Suresh and Praneet Srivastava who played a central role in motivating me and helping me discover the beauty of mathematics. I also wish to thank Medhini Sridharr, Simran Prashar and Somya Bansal for being the best kind of friends I could ask for and for keeping up with all my antics and moody personality. I will always remember to eat thanks to you.

Finally, I would like to thank my family for always being there for me ever since I was a restless kid who could only be found on tree tops up to now, a (still immature) twenty two year old. Firstly, I would like to thank Pachuppan for all his guidance and strong involvement in the many choices I made relating to my academic career. I would like to thank Ettan and Shubhopol, for giving me a place I could take a break and enjoy the weekend during my time at Chennai. I would also like to thank Kunju and Ammini for all the fun we had a jokes we cracked and all the lovely YouTube channels you introduced me to. I would also like to thank Vallyoppol and Sreekanthettan for helping me take break during my weekends in Chennai and also for all the amazing late night movies, birthdays and other adventure we had in Delhi. Also to Pachu for all fights, games and horror movies we enjoyed in Delhi and in Chennai. It would be impossible not to thank my sister for being there since I was a child and for being someone who wouldn't judge for whatever I did or said. I hope we keep fighting even when we are in our eighties. Lastly, but most importantly, I would like to thank my parents who have played the most important part in all my progress in life. I am thankful that they always let me make my own decisions in life and supported it no matter what. They have played the most significant part in helping me stand on my own two feet and learn from my own mistakes.

Contents

Chapter 1

Prerequisites

This chapter will mainly contain most of the material that would be required to to understand future content. Although it will cover most of the material and the reader could start of with no knowledge of the content in this chapter, that is not recommended as this thesis will quickly brush through topics that even the best books spend an entire chapter explaining. Lemmas, theorems, propositions and corollaries stated in this chapters will not be proven, as that is not the intent of this thesis and they can be found in most texts written on these topics.

1.1 Commutative Algebra

This section will introduce definitions and results required from Commutative Algebra for the rest of the text. It will not contain all results discussed in a standard course in Commutative Algebra. [1] and [4] are good references for most proofs.

1.1.1 Rings

Definition. A ring A is a set with two binary operations (addition and multiplication) such that:

1. A is an abelian group with respect to addition. The additive identity is denoted by 0 and for any $x \in A$, the additive inverse is denoted by $-x$.

2. Multiplication is associative and distributive over addition.

3. Multiplication is commutative.

4. There exists a $1 \in A$, such that $x1 = 1x = x$

It is import to note that it is possible to have rings where $1 = 0$. In fact in the zero ring $\{0\}$, $1 = 0$. One can later show that this is the only such ring with this property (up to isomorphism).

Definition. For rings A and B, a $\mathsf{ring\ homomorphism}$ (from A to B) is a function $f : A \to B$ such that for all $x, y \in A$:

1. $f(x + y) = f(x) + f(y)$

2. $f(xy) = f(x)f(y)$
3. $f(1) = 1$

An $\mathtt{isomorphism}$ is a bijective ring homomorphism. A $\mathtt{subring}$ of a ring A is a non empty subset $S \subseteq A$ containing 1 such that S is a ring under the same operations.

Definition. An ideal of a ring A is a non empty subset $\mathfrak{a} \subseteq A$ such that $x + y \in \mathfrak{a}$ for all $x, y \in \mathfrak{a}$ and $rx \in \mathfrak{a}$ for all $x \in \mathfrak{a}$ and $r \in A$.

Definition. A proper ideal $\mathfrak{p} \subseteq A$ is said to be $\mathtt{prime}$ if whenever $xy \in \mathfrak{p}$, then either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

Proposition 1.1. *Given any homomorphism* $f : A \to B$ *or rings and prime ideal* $\mathfrak{p} \subseteq B$, $f^{-1}(\mathfrak{p})$ *is a prime ideal of* A.

Definition. A proper ideal $\mathfrak{m} \subseteq A$ is said to be $\mathtt{maximal}$ if for any ideal $\mathfrak{a} \subseteq A$, if $\mathfrak{m} \subseteq \mathfrak{a}$, then either $\mathfrak{a} = \mathfrak{m}$ or $\mathfrak{a} = A$.

Definition. Let A be a ring and $\mathfrak{a} \subseteq A$ an ideal. Denote by $A/\mathfrak{a}$, the set of equivalence classes of A under the equivalence relation where $x \in A$ is related to $y \in A$ if and only if $x - y \in \mathfrak{a}$. Give $A/\mathfrak{a}$ a ring structure be defining $[x] + [y] = [x + y]$ and $[x][y] = [xy]$. It is easy to check that the above relation is in fact and equivalence relation and that the two binary operations on $A/\mathfrak{a}$ does satisfy the properties of a ring. The ring $A/\mathfrak{a}$ is called a $\mathtt{quotient\ ring}$ of A received by quotienting A by $\mathfrak{a}$.

For any ideal $\mathfrak{a} \subseteq A$, there is a natural surjective ring homomorphism $f : A \to A/\mathfrak{a}$ which takes each element to its equivalence class. In fact the next proposition says that these are the only kinds of ring homomorphism.

Proposition 1.2. *For any surjective ring homomorphism* $f : A \to B$, $A/\ker(f) \cong B$, *where* $\ker(f) = \{x \in A | f(x) = 0\}$ *is called the kernel of* f.

Proposition 1.3. *For any ideal* $\mathfrak{a} \subseteq A$, *the ideal of* $A/\mathfrak{a}$ *are in one-to-one correspondence with the ideals of* A *containing* $\mathfrak{a}$. *This correspondence can be established with (and only) prime ideals going to prime ideals and with (and only) maximal ideal going to maximal ideal.*

Definition. An element $x \in A$ is said to be a zero divisor if there exists a non-zero element $y \in A$ such that $xy = 0$.

Definition. A ring is said to be an $\mathtt{integral\ domain}$ if 0 is the only zero divisor and $1 \neq 0$. Sometimes an integral domain is just referred to as a domain.

Definition. An element $u \in A$ is said to be a $\mathtt{unit}$ if there exists a $v \in A$ such that $uv = 1$.

Definition. A ring is said to be a $\mathtt{field}$ if every non-zero element is a unit and $1 \neq 0$.

Proposition 1.4. *Let* $\mathfrak{p} \subseteq A$ *be an ideal. Then,* $\mathfrak{p}$ *is prime if and only if* $A/\mathfrak{p}$ *is a domain.*

Proposition 1.5. *Let* $\mathfrak{m} \subseteq A$ *be an ideal. Then,* $\mathfrak{m}$ *is maximal if and only if* $A/\mathfrak{m}$ *is a field.*

Proposition 1.6. *Every field is an integral domain.*

Proposition 1.7. *Every maximal ideal is prime.*

Definition. For a given domain A, the $\mathtt{field\ of\ fraction}$ of A is the smallest field containing A (up to isomorphism).

Definition. An element $n \in A$ is said to be $\mathtt{nilpotent}$ if there exists a $k \in \mathbb{N}$ such that $n^k = 0$. The set of all nilpotent elements form an ideal $\mathrm{nil}(A)$ called the $\mathtt{nilradical}$ of A.

Proposition 1.8. *The nilradical of A is equal to the intersection of all prime ideals.*

Definition. The intersection of all maximal ideal of A if called the $\mathtt{Jacobson\ radical}$ of A

Proposition 1.9. *For any x in the Jacobson radical of A and any $y \in A$, $1 - xy$ is a unit.*

Definition. For and ideal $\mathfrak{a} \subseteq A$, the $\mathtt{radical\ of}$ $\mathfrak{a}$ denoted by $\mathrm{rad}(\mathfrak{a})$ or $\sqrt{\mathfrak{a}}$ is the ideal comprising of all those elements $x \in A$ such that $x^n \in \mathfrak{a}$ for some $n \in \mathbb{N}$. An ideal is said to be a $\mathtt{radical\ ideal}$ if $\mathrm{rad}(\mathfrak{a}) = \mathfrak{a}$.

Proposition 1.10. *For an ideal $\mathfrak{a} \subseteq A$, if S is the set of all prime ideal containing $\mathfrak{a}$, then, $\mathrm{rad}(\mathfrak{a}) = \bigcap_{\mathfrak{p} \in S} \mathfrak{p}$.*

Proposition 1.11. *Both, the nilradical and Jacobson radical are radical ideals.*

Definition. Given a prime ideal $\mathfrak{p} \subseteq A$, we define the $\mathtt{height}$ of $\mathfrak{p}$ to the supremum of all natural numbers n such that there exists a chain of prime ideals $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n = \mathfrak{p}$. The $\mathtt{dimension}$ of a ring is the supremum of heights of every prime ideals in a ring.

Proposition 1.12. *Krull's Hauptidealsatz. Let A be a noetherian ring, and let $f \in A$ be an element which is neither a zero divisor not a unit. Then every minimal prime ideal containing f has height 1.*

Definition. Given a ring A, we denote by $A[x]$ the ring of polynomials with coefficients in A. We say that a field k is $\mathtt{algebraically\ closed}$ if every polynomial in $k[x]$ has a zero.

Definition. Given a ring A and a subring ring B, we define the $\mathtt{integral\ closure}$ of B over A to be the all elements of A which are the root of some monic polynomial over B. We say that B is $\mathtt{integrally\ closed\ over}$ A if its integral closure over A is itself.
We say that a domain is $\mathtt{integrally\ closed}$ if it is integrally closed over its field of fractions.

1.1.2 Modules

Definition. Given a ring A, a $\mathtt{module}$ over A is an abelian group M such that there if a function $f : A \times M \to M$ ($f(x, m)$ is denoted by xm) which satisfies the following properties for all $x, y \in A$ and $m, n \in M$:

1. $x(m + n) = xm + xn$
2. $(x + y)m = xm + ym$
3. $(xy)m = x(ym)$
4. $1m = m$

A module over a ring A is referred to as an A-module.

Note that a modules are a generalization of vector spaces over a field. It is important to note that every abelian group has a natural $\mathbb{Z}$-module structure on it. Any A-module M is said to be $\mathsf{finitely\ generated}$ if there exists elements $m_1, ..., m_k \in M$ such that any $m \in M$ can be written as $m = x_1m_1 + \cdots + x_km_k$ where $x_1, ..., x_k \in A$. Any module that is not finitely generated is called infinitely generated.

Definition. Given two A-modules, M and N, an A-module homomorphism from M to N is a function $f : M \to N$ such that for all $a \in A$ and $m, n \in M$:

$$f(am) = af(m)$$

$$f(m + n) = f(m) + f(n)$$

An A-module homomorphism is also sometimes referred to as an A-linear map. A bijective module homomorphism is an isomorphism.

$\mathrm{Hom}_A(M, N)$ is the set of all A-module homomorphisms from M to N and this has a natural A-module structure.

Definition. Given an A-module module M, a $\mathsf{submodule}$, is a subgroup $M' \subseteq M$ such that $am \in M'$ for all $a \in A$ and $m \in M'$. A $\mathsf{quotient\ module}$ is a quotient group ${}^{M}/_{M'}$ with its natural module structure.

Definition. Given an A-module homomorphism is $f : M \to N$, the kernel is $\mathrm{ker}(f) = \{m \in M | f(m) = 0\}$ and the $\mathsf{cokernel}$ is $\mathrm{coker}(f) = {}^{N}/_{\mathrm{Img}(f)}$, where $\mathrm{Img}(f)$ is the image of f.

Proposition 1.13. *1. A module homomorphism is injective if and only if the kernel is* 0.

2. A module homomorphism is surjective if and only if the cokernel is 0.

Proposition 1.14. *1. if* $N \subseteq M \subseteq L$ *are* A*-modules, then*

$$({}^{L}/_{N})/({}^{M}/_{N}) \cong {}^{L}/_{M}$$

2. If M_1 *and* M_2 *are submodule of* M*, then,*

$$({}^{M_1+M_2})/_{M_1} \cong {}^{M_2}/_{(M_1 \cap M_2)}$$

Given a family of A-modules $\{M_i\}_{i \in I}$, we define the direct sum of all the modules as a family of elements $(x_i)_{i \in I}$, where each $x_i \in M_i$ and all but finitely many of these are zero. The collection of such elements along with the A-module structure (coordinate-wise addition) on them is called the direct sum of the family and denote by $\bigoplus_{i \in I} M_i$. If the set I is finite, then there is no difference between the direct sum and usual product.
Now, for any ideal $a \subseteq A$ and A-module M we introduce the notation aM to be the submodule consisting of elements of the form $\Sigma_{i=1}^{n} x_i m_i$ where $x_i \in a$ and $m_i \in M$.

Proposition 1.15. *Nakayama's lemma. Let* M *be a finitely generated* A*-module and* $a \subseteq A$ *an ideal contained in the Jacobson radical of* A*. Then,* $aM = M$ *implies that* $M = 0$.

Corollary 1.16. *Let* M *be a finitely generated* A*-module,* N *a submodule of* M *and* $a \subseteq A$ *contained in the Jacobson radical. Then* $M = aM + N$ *implies that* $M = N$.

An sequence of modules

$$\cdots \xrightarrow{f_{i-1}} M_{i-1} \xrightarrow{f_i} M_i \xrightarrow{f_{i+1}} M_{i+1} \xrightarrow{f_{i+2}} \cdots$$

is said to be exact at M_i if $\mathrm{img}(f_i) = \ker(f_{i+1})$. A sequence which is exact at every module in the sequence is said to be an exact sequence.
A short exact sequence is an exact sequence of the form

$$0 \to M' \to M \to M'' \to 0$$

In Particular,
$0 \to M' \xrightarrow{f} M$ is exact if and only if f is injective;
$M \xrightarrow{f} M'' \to 0$ is exact if and only if f is surjective.

Definition. An A-module M is said to be $\mathtt{free}$ if there exists a set I such that $M \cong \bigoplus_{i \in I} A$ as A-modules.

Definition. Given a free A-module M, we define the $\mathtt{dual}$ of M, $M^* = \mathrm{Hom}_A(M, A)$

Proposition 1.17. *Let M be a finitely generated free A-module. Then, there is a natural isomorphism $M \cong M^{**}$*

Definition. For a given A-modules M and N, $M \times N$ is also an A-module. A bilinear map from $M \times N$ to an A-module P is a function $f : M \times N \to P$ such that the maps $f_m = f(m, \cdot) : N \to P$ and $f_n = f(\cdot, n) : M \to P$ for any $m \in M$ and $n \in N$ are linear.

Definition. Let $f : M \times N \to A$ be a bilinear map where M and N are both free modules. We say that f is a $\mathtt{perfect\ pairing}$ if the natural maps $M \to N^*$ and $N \to M^*$ are both isomorphisms.

Definition. Given A-modules M and N, the $\mathtt{tensor\ product}$ of M and N is a module $M \bigotimes_A N$ along with a bilinear map $f : M \times N \to M \bigotimes_A N$ such that for any A-module P and bilinear map $f' : M \times N \to P$, there is a unique linear map $g : M \bigotimes_A N \to P$ such that $g \circ f = f'$

Proposition 1.18. *The definition of tensor product gives a canonical isomorphism $\mathrm{Hom}(M \bigotimes_A N, P) \cong \mathrm{Hom}(M, \mathrm{Hom}(N, P))$.*

Given any two A-modules M and N, the tensor product of M and N exists and that module is denoted by $M \bigotimes_A N$ or $M \bigotimes N$ depending on whether the ring need to be mentioned or not and by abuse of notation the image of any (m, n) in $M \bigotimes N$ is denoted by $m \bigotimes n$. Moreover, the following tensor products are all isomorphic:

1. $M \bigotimes N \cong N \bigotimes M$.
2. $(M \bigotimes N) \bigotimes P \cong M \bigotimes (N \bigotimes P)$.
3. $(M \bigoplus N) \bigotimes P \cong (M \bigoplus P) \bigotimes (N \bigoplus P)$.
4. $A \bigotimes_A M \cong M$

Let A and B be ring and $f : A \to B$ be a ring homomorphism. Then, any B-module M, has a natural A-module structure induced by f, i.e. $am = f(a)m$ where $a \in A$ and $m \in M$. In particular, B has an A-module structure.

Definition. Let A and B be rings. Let $f : A \to B$ be a ring homomorphism. This homomorphism equips B with an A-module structure. B, equipped with this induced A-module structure is called an $\mathtt{A\text{-}algebra}$.

Given an A-algebra B and a A-module M, we define $M_B = B \bigotimes_A M$ which has a natural B-module structure with $b'(b \bigotimes m) = (b'b) \bigotimes m$.
Given two A-algebras B and C, $B \bigotimes C$ also has a very natural A-algebra structure which can be verified through some basic diagram chasing.

Definition. Given a field k and another field K such $k \subseteq K$. Then K has a natural k-vector space (in fact k-algebra) structure. An element $a \in K$ is said to be $\mathtt{algebraic}$ over k if there exists a polynomial $p(x)$ over k such that $p(a) = 0$. An elements that is not algebraic is called $\mathtt{transcendental}$. A set $S \subseteq K$ is said to be $\mathtt{algebraically\ independent}$ if for any finite number of elements $s_1, ..., s_n \in S$ the natural map $k[x_1, ..., x_n] \to k[s_1, ..., s_n]$ is injective. Here $k[s_1, ..., s_n]$ is the smallest k-subalgebra of K which contains $s_1, ..., s_n$. A maximal algebraically independent set is called a $\mathtt{transcendental\ basis}$. The $\mathtt{transcendence\ degree}$ is the cardinality of the transcendental basis.

Proposition 1.19. *Let k be a field, and let B be an integral domain which is finitely generated k-algebra. Then:*

1. *the dimension of B is equal to the transcendence degree of the quotient field $K(B)$ of B over k.*
2. *For any prime ideal p in B, we have $\mathrm{height}(p) + \dim B/p = \dim B$.*

1.1.3 Localisation

Definition. Given a ring A, a subset $S \subseteq A$ is said to be a $\mathtt{multiplicative\ set}$ if:

1. $1 \in S$
2. For all $x, y \in S$, $xy \in S$.

Let A be a ring and $S \subseteq A$ be a multiplicative set. Consider the set $A \times S$ and an equivalence relation $\sim$ on it. $(x, s_1) \sim (y, s_2)$ if there exists a $t \in S$ such that $t(xs_2 - ys_1) = 0$. Let $S^{-1}A$ denote the set of equivalence classes and denote the equivalence class of some (a, s) as a/s. One can check that in fact, $S^{-1}A$ has a natural ring structure. There is natural ring homomorphism $i : A \to S^{-1}A$ where $i(a) = a/1$. Note that i in general need not be injective. In particular, given a prime ideal $p \subseteq A$, let $S = A \setminus p$ and we denote $S^{-1}A$ by A_p and call it the $\mathtt{localisation}$ of A at p. And, given an element $f \in A$, let $T = \{f^n | n \in \mathbb{N}\} \cup \{1\}$ and denote $T^{-1}A$ by A_f and call it the $\mathtt{localisation}$ of A at f.

Proposition 1.20. *Universal Mapping Property of Localisation. Given a ring homomorphism $f : A \to B$ and a multiplicative set $S \subseteq A$ such that $f(s)$ is a unit for all $s \in S$, there exists a unique homomorphism from $f' : S^{-1}A \to B$ such that $f' \circ i = f$. Moreover, the map f' is an isomorphism if and only if the following two conditions hold:*

1. $f(a) = 0$ *implies that* $as = 0$ *for some* $s \in S$.

2. *For any* $b \in B$*, there exists* $a \in A$ *and* $s \in S$ *such that* $b = f(a)f(s)^{-1}$.

Given an A-module M, one can define $S^{-1}M$ in a similar manner and one can check that this forms an $S^{-1}A$-module and A-linear maps $M \xrightarrow{f} N$ give us $S^{-1}A$-linear maps $S^{-1}M \xrightarrow{S^{-1}f} S^{-1}N$. We use the notation $M_\mathfrak{p}$ and M_f in the same situations as rings ($\mathfrak{p} \subseteq A$ is a prime ideal and $f \in A$).

Proposition 1.21. S^{-1} *is exact. Given an exact sequence* $M' \xrightarrow{f} M \xrightarrow{g} M''$ *of* A*-module and a multiplicative set* $S \subseteq A$*, we get an exact sequence* $S^{-1}M' \xrightarrow{S^{-1}f} S^{-1}M \xrightarrow{S^{-1}g} S^{-1}M''$ *of* $S^{-1}A$ *modules.*

Proposition 1.22. *Given an* A*-module* M *and a multiplicative set* $S \subseteq A$*, there is a unique isomorphims form* $S^{-1}A \bigotimes M$ *to* $S^{-1}M$ *of* $S^{-1}A$*-modules.*

A property $\mathcal{P}$ of A (or an A-module M) is said to be local if:
A (or M) has $\mathcal{P} \iff A_\mathfrak{p}$ (or $M_\mathfrak{p}$) has $\mathcal{P}$ for all prime ideal $\mathfrak{p} \subseteq A \iff A_\mathfrak{m}$ (or $M_\mathfrak{m}$) has $\mathcal{P}$ for all maximal ideal $\mathfrak{m} \subseteq A$.

Proposition 1.23. *The following are local properties:*

1. $M = 0$.

2. *homomorphism is injective.*

3. *homomorphism is surjective.*

Proposition 1.24. *Every ideal in* $S^{-1}A$ *is generated by some* $i(\mathfrak{a})$ *for some ideal* $\mathfrak{a} \subseteq A$ *and this is represented as* $S^{-1}\mathfrak{a}$.

Proposition 1.25. *For a ring* A *and a multiplicative set* $S \subseteq A$*, the prime ideal of* $S^{-1}A$ *are in order preserving bijection with prime ideal of* A *that do not intersect with* S.

1.1.4 Types of Rings and Modules

Definition. A ring A is said to be local if it has a unique maximal ideal. If $\mathfrak{m} \subseteq A$ is its unique maximal ideal, it is sometimes denoted as $(A, \mathfrak{m})$

Proposition 1.26. *For any prime ideal* $\mathfrak{p} \subseteq A$*,* $A_\mathfrak{p}$ *is a local ring.*

Definition. Given local rings $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$, a homomorphism $f : A \to B$ is said to be a $\mathsf{local\ homomorphism}$ if $f^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A$.

Definition. In a ring A, the following conditions are equivalent:

1. Every ideal if finitely generated.

2. Any collection of ideal has a maximal element.

3. Any chain of ideals terminate.

Rings satisfying any of the above properties are called a $\mathsf{Noetherian\ ring}$.

Proposition 1.27. *Hilbert's Basis Theorem. If* A *is a noetherian ring, then* $A[x]$ *is noetherian.*

Corollary 1.28. *If* A *is a noetherian ring, then for any* $n \in \mathbb{N}$, $A[x_1, ..., x_n]$ *is noetherian.*

Given a totally ordered abelian group G and a field K, a map $\nu : K \setminus \{0\} \to G$ is said to be a valuation on K if for any $x, y \in K \setminus \{0\}$:

1. $\nu(xy) = \nu(x) + \nu(y)$.
2. $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$

The valuation ring of K received from ν is the ring $A = \{x \in K | \nu(x) \geq 0\} \cup \{0\}$.

Definition. A valuation ring is an integral domain that is result of a valuation of on its field of fractions. The valuation ring becomes a discrete valuation ring if the totally ordered group is $\mathbb{Z}$.

Proposition 1.29. *A valuation ring is a local ring with the unique maximal ideal being* $\{x \in K | \nu(x) > 0\} \cup \{0\}$.

Definition. Let A be a noetherian local ring with the maximal ideal $\mathfrak{m}$ and residue field $k = A/\mathfrak{m}$. We say that A is a regular local ring if $\dim_k \mathfrak{m}/\mathfrak{m}^2 = \dim A$.

Definition. Let M be an A-module. Then we say that a sequence of elements $x_1, ..., x_n$ in A is regular in M if x_1 is non-zero divisor in M and for each $i = 2, ..., n$, x_i is a non-zero divisor in $M/(x_1,...,x_{i-1})M$. We say that an $x \in A$ is a non-zero divisor in M if for any $m \in M$, if $xm = 0$, then $m = 0$. If $(A, \mathfrak{m})$ is a local ring and M an A-module, then we define the depth of M to be the largest n such that there is a sequence of n elements in A which are regular in M.

Proposition 1.30. *Let* M *be an* A*-module and* I *some indexing set. Then,* $\operatorname{depth} M = \operatorname{depth} \bigoplus_{i \in I} M$

Proposition 1.31. *Given a ring* A*,* $\dim A \geq \operatorname{depth}_A A$

Definition. A noetherian local ring A is called Cohen-Macaulay if $\operatorname{depth} A = \dim A$.

Proposition 1.32. *Let* A *be a local noetherian ring. If* A *is regular, then* A *is Cohen-Macaulay.*

Proposition 1.33. *Let* A *be a Cohen-Macaulay noetherian local ring and let* $\mathfrak{p} \subseteq A$ *be a prime ideal. Then,* $A_\mathfrak{p}$ *is also a Cohen-Macaulay noetherian local ring.*

Definition. A graded ring is a ring S together with a decomposition $S = \bigoplus_{d \geq 0} S_d$. such that for any $d, e \geq 0$, $S_d \cdot S_e \subseteq S_{d+e}$. An element of S_d for some $d \geq 0$ is called a homogeneous element of degree d. An ideal $\mathfrak{a} \subseteq S$ is said to be a homogeneous ideal if $\mathfrak{a} = \bigoplus_{d \geq 0}(S_d \cap \mathfrak{a})$ (i.e. $\mathfrak{a}$ can be generated by homogeneous elements).

Proposition 1.34. *The sum, product, intersection, and radical of homogeneous ideal is homogeneous.*

Proposition 1.35. *Let* $\mathfrak{p} \subseteq S$ *be a homogeneous ideal. Then,* $\mathfrak{p}$ *is prime if and only if for any homogeneous* $f, g \in S$*, if* $fg \in \mathfrak{p}$*, then either* $f \in \mathfrak{p}$ *or* $g \in \mathfrak{p}$.

Definition. For a homogeneous ring S, let $\mathfrak{p} \subseteq S$ be a prime ideal and $f \in S$. Define $S_{(\mathfrak{p})}$ be the subring of the localization $S_\mathfrak{p}$ of elements of degree 0 and $S_{(f)}$ the subring of S_f of elements of degree 0. we say an element is of degree zero if it is a quotient of homogeneous elements, both of which have the same degree.

Proposition 1.36. *Given a graded ring S, we define a graded S-module M to be an S-module with a decomposition $M = \bigoplus_{d\in\mathbb{Z}} M_d$ such that for any $d \geq 0$ and $e \in \mathbb{Z}$, $S_d \cdot M_e \subseteq M_{d+e}$.*
Given a graded S-module M and an $n \in \mathbb{Z}$, we define a new graded S-module $M(n) = \bigoplus_{d\in\mathbb{Z}} N_d$, where $N_d = M_{d+n}$.
Given graded S-modules $M = \bigoplus_{d\in\mathbb{Z}} M_d$ and $N = \bigoplus_{d\in\mathbb{Z}} N_d$, we define a graded homomorphism $f : M \to N$ to be an S-module homomorphism such that $f(M_d) \subseteq N_d$.

Definition. For any A-module M , we define $T^n(M) = \bigotimes_{i=1}^n M$ and we define the A-algebra $T(M) = \bigoplus_{n\geq 0} T^n(M)$ where $T^0(M) = A$ (Note that $T(M)$ satisfies all properties of a ring except maybe commutativity). Now we define the $\mathsf{exterior\ direct\ product\ of}\ M$, $\bigwedge(M) = \bigoplus_{n\geq 0} \bigwedge^n(M)$ which is $T(M)$ quotiented by the two sided ideal generated by all elements of the form $m \bigotimes m$.

1.1.5 Kähler Differentials

Definition. Let A be a ring and B and A-algebra and M be a B-module. An A-derivation of B into M is a groups homomorphisms $d : B \to M$ such that $d(bb') = b \cdot d(b') + b' \cdot d(b)$ for all $b, b' \in B$ and $d(a \cdot 1) = 0$ for all $a \in A$.

Definition. Let A be a ring and B an A-algebra. Then we define the $\mathsf{module\ of\ relative\ differentials}$ (or $\mathsf{module\ of\ Kähler\ differentials}$) of B over A, $\Omega_{B/A}$ together with an A-derivation $d : B \to \Omega_{B/A}$ such that for any B-module M and A-derivation $d' : B \to M$, there exists a B-module homomorphism $f : \Omega_{B/A} \to M$ such that $f \circ d = d'$

Proposition 1.37. *Let B be an A-algebra. Let $f : B \bigotimes_A B \to B$ be the homomorphism $f(b \bigotimes b') = bb'$. Let I bet the kernel of f. Now $B \bigotimes B$ can be considered as a B-module by multiplication on the left. Thus I/I^2 inherits the B-module structure. Let $d : B \to I/I^2$ be the function $d(b) = (1 \bigotimes b) - (b \bigotimes 1)$ modulo I^2. Then, the pair I/I^2 with the function d forms the module of Kähler differentials of B over A.*

Proposition 1.38. *let A' and B be A-algebras and $B' = B \bigotimes_A A'$. Then, $\Omega_{B'/A'} \cong \Omega_{B/A} \bigotimes_B B'$. Furthermore, If S is a multiplicative system in B, then $\Omega_{S^{-1}B/A} \cong S^{-1}\Omega_{B/A}$.*

Proposition 1.39. *Let $A \to B \to C$ be a sequence of rings. Then there is a natural exact sequence of C-modules: $\Omega_{B/A} \bigotimes_B C \to \Omega_{C/A} \to \Omega_{C/B} \to 0$*

Proposition 1.40. *Let B be an A-algebra, let I be an ideal of B and let $C = B/I$. Then there is a natural exact sequence of C-modules: $I/I^2 \xrightarrow{\delta} \Omega_{B/A} \bigotimes_B C \to \Omega_{C/A} \to 0$. Here, $\delta(\overline{b}) = db \bigotimes 1$*

Proposition 1.41. *If B is a finitely generated A-algebra, then $\Omega_{B/A}$ is a finitely generated B-module.*

1.2 Basic Topology

Much like the previous section, this will only introduce definitions and results that are necessary for understanding the rest of the thesis. Infact, we will not discuss topics like Hausdorffness, path homotopy, etc. which are very fundamental to a course in topology.

Definition. A topological space is an ordered pair $(X, \mathscr{T}_X)$ where X is a set and $\mathscr{T}_X$ is a subset of the powerset of X such that:

1. $\emptyset, X \in \mathscr{T}_X$.
2. for any family of sets $\{X_\lambda\}_{\lambda\in\Lambda}$ in $\mathscr{T}_X$, $\bigcup_{\lambda\in\Lambda} X_\lambda \in \mathscr{T}_X$.
3. For any finite collection of elements $X_1, ..., X_n$ in $\mathscr{T}_X$, $X_1 \cap \cdots \cap X_n \in \mathscr{T}_X$.

The set $\mathscr{T}_X$ is called a topology on X and elements of the topology are called $\mathtt{open\ sets}$. A subset $W \subseteq T$ is said to be $\mathtt{closed\ set}$ if $W^c \in \mathscr{T}_X$.
The notation $(X, \mathscr{T}_X)$ is usually replaced by just writing a 'A topological space X' where the set $\mathscr{T}_X$ is understood based on the description.

Proposition 1.42. *For a given topological space* X.

1. $\emptyset$ *and* X *are both closed*
2. *For any given family of closed subset* $\{W_\lambda\}_{\lambda\in\Lambda}$ *in* X, $\bigcap_{\lambda\in\Lambda} W_\lambda$ *is closed.*
3. *If* $W_1, ..., W_n$ *are closed subsets in* X, *then* $W_1 \cup \cdots \cup W_n$ *is also closed.*

A topological space could also be defined by specifying the closed sets rather than the open sets.
Given a subset $Z \subseteq X$, The closure of Z (denoted by $\overline{Z}$) is the intersection of all closed sets containing Z. This is also the smallest closed set containing Z, i.e. if any closed set W contains Z, then $\overline{Z} \subseteq W$. A set is said to be $\mathtt{dense}$ if $\overline{Z} = X$
A subset $\mathcal{B} \subseteq \mathscr{T}_X$ is said to be $\mathtt{base\ of\ the\ topology}$ if any element $U \in \mathscr{T}_X$ such that $U \neq \emptyset$ can be written as a union of elements in $\mathcal{B}$. Elements of $\mathcal{B}$ are called $\mathtt{basic\ open\ sets}$.

Definition. Given a topological space X and a subset $Y \subseteq X$, we give Y subspace topology by assigning open sets to be sets of the form $U \cap Y$ where U is open in X. It can be easily checked that sets of this form satisfy the properties defined for an open set. The topological space Y is said to be a $\mathtt{subspace}$ of X.

Definition. A topological space X is said to be $\mathtt{irreducible}$ if there does not exists proper closed subsets Y_1 and Y_2 such that $X = Y_1 \cup Y_2$. A subset $Z \subseteq X$ is said to be irreducible if it is irreducible as a subspace.

Proposition 1.43. *An topological space is irreducible if and only any two non-empty open sets intersect (which is equivalent to saying that every open set is dense).*

Proposition 1.44. *Every open subset of an irreducible topological space is irreducible.*

Proposition 1.45. *Given an irreducible subset $Y \subseteq X$, then $\overline{Y}$ is also irreducible*

Proposition 1.46. *The closure of a single point (singleton set containing that point) is irreducible.*

Definition. Given a topological space X an $\mathtt{open\ cover}$ of that space is a family of open subsets $\{V_\lambda\}_{\lambda\in\Lambda}$ such that $\bigcup_{\lambda\in\Lambda} V_\lambda = X$. The space X is said to be $\mathtt{quasi\text{-}compact}$ if, for any open cover $\{V_\lambda\}_{\lambda\in\Lambda}$, there exists a finite sub-cover, i.e. finitely many open sets in the same family $V_{\lambda_1}, ..., V_{\lambda_n}$ such that $V_{\lambda_1} \cup \cdots \cup V_{\lambda_n} = X$.

In most topology books they use the term compact instead of quasi-compact, but we shall stick we the word quasi-compact.

Definition. A topological space is said to be $\mathtt{noetherian}$, if every descending chain of closed sets terminate.

Proposition 1.47. *In a noetherian topological space, every nonempty closed set Y has a unique finite covering of closed irreducible sets $Y_1, ..., Y_k$ (i.e. $Y = Y_1 \cup \cdots \cup Y_k$) such that $Y_i \subseteq Y_j$ if and only if $i = j$.*

Definition. Given a topological spaces X and Y, a function $f : X \to Y$ is said to be continuous if for any open subset $U \subseteq Y$, $f^{-1}(U)$ is open.

Proposition 1.48. *A function between two topological spaces is continuous if and only if inverse of a closed set is closed*

Definition. A continuous function $f : X \to Y$ is said to be a $\mathtt{homeomorphism}$ if it is bijective and the inverse function $f^{-1} : Y \to X$ is also continuous.

Definition. Given a topological space X and an equivalence relation $\sim$ on the set X. Let ${}^{X}\!/\!_{\sim}$ be the set of equivalence classes of X under $\sim$. Let $p : X \to {}^{X}\!/\!_{\sim}$ be the natural projection function. Now define a topology on ${}^{X}\!/\!_{\sim}$ by defining those open sets to be those sets U such that $p^{-1}(U)$ is open. This topology is said to be a $\mathtt{quotient\ topology}$ and we call ${}^{X}\!/\!_{\sim}$ to be a $\mathtt{quotient\ space}$.

Definition. Given a topological space, the $\mathtt{dimension}$ of the same is defined to be set supremum of all those n such there exists a sequence of irreducible closed sets $Z_0 \subsetneq Z_1 \subsetneq \cdots \subsetneq Z_n$. If $Y \subseteq X$ is a subspace, we define the $\mathtt{codimension}$ of Y with respect to X to be $\dim X - \dim Y$.

Definition. We say that a topological space is $\mathtt{equidimensional}$ if all its irreducible components (i.e. maximal irreducible closed sets) are of the same dimension.

1.3 Category Theory

Definition. A $\mathtt{category}$ is a collection of $\mathtt{object}$ and $\mathtt{morphisms}$ between between objects such that:

1. Given any morphism f from object A to B (denoted by $f : A \to B$) and morphism g from object B to C (denoted by $g : B \to C$), there is a unique morphism from A to C denoted by $g \circ f$.

2. For any object A, there exists a morphism $\mathrm{id} : A \to A$ such that for any morphism $f : A \to B$ and morphism $g : C \to A$, $f \circ \mathrm{id}_A = f$ and $\mathrm{id}_A \circ g = g$

3. For morphisms $h : A \to B$, $g : B \to C$ and $f : C \to D$, $(f \circ g) \circ h = f \circ (g \circ h)$

The following are important categories that shall used in the this book:

1. The category **Sets** where objects are sets and morphisms are functions between sets.

2. The category **Grp** where objects are groups and morphisms are group homomorphisms.

3. The category **Ab** where objects are abelian groups and morphisms are group homomorphisms.

4. The category **Rings** where objects are rings and morphisms are ring homomorphisms.

5. The category **Top** where objects are topological spaces and morphisms are continuous functions.

6. For a topological space X, the category $\mathbf{Top(X)}$ where objects are open sets and morphisms are inclusion maps.

Definition. An $\mathtt{isomorphism}$ is a morphism $f : A \to B$ such that there exists a morphism $g : B \to A$ such that $f \circ g = \mathrm{id}_B$ and $g \circ f = \mathrm{id}_A$. If there exists an isomorphisms between two objects, the two objects are said to be $\mathtt{isomorphic}$

Definition. Let $\mathcal{C}$ be a category. for any two objects A and B, we denote by $\mathrm{Hom}_{\mathcal{C}}(A, B)$ (or simply $\mathrm{Hom}(A, B)$) as the collection of morphisms between A and B. The category is said to be $\mathtt{locally\ small}$ if $\mathrm{Hom}(A, B)$ is a set for any pair of objects A and B.

Definition. An object I is called the $\mathtt{initial\ object}$ of a category if for any object A, there exists a unique morphism $f : I \to A$. An object F is said to be a $\mathtt{final\ object}$ if for any object B, there exists a unique morphism $g : B \to F$.

Proposition 1.49. *The initial and final objects if exist, are unique up to unique isomorphism.*

Definition. Given two categories $\mathcal{C}$ and $\mathcal{D}$ a $\mathtt{functor}$ $\mathcal{F}$ between the categories (denoted as $\mathcal{F} : \mathcal{C} \to \mathcal{D}$) assigns an object to an object ($A \mapsto \mathcal{F}(A)$) and morphism to a suitable morphism. There are two kinds of functors:

1. $\mathtt{Covariant\ functors}$. The following are the conditions it must satisfy:
   (a) $(f : A \to B) \mapsto \mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$
   (b) $\mathcal{F}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{F}(A)}$ for any object A in $\mathcal{C}$.
   (c) if $g : A \to B$ and $f : B \to C$ are morphisms in $\mathcal{C}$, then $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$

2. $\mathtt{Contravariant\ functor}$. The following are the conditions it must satisfy:
   (a) $(f : A \to B) \mapsto \mathcal{F}(f) : \mathcal{F}(B) \to \mathcal{F}(A)$
   (b) $\mathcal{F}(\mathrm{id}_A) = \mathrm{id}_{\mathcal{F}(A)}$ for any object A in $\mathcal{C}$.
   (c) if $g : A \to B$ and $f : B \to C$ are morphisms in $\mathcal{C}$, then $\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f)$

Definition. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a covariant functor between locally small categories. We say:

1. $\mathcal{F}$ is faithful if the induced function $\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(A), \mathcal{F}(B))$ is injective for every pair of objects A and B in $\mathcal{C}$.

2. $\mathcal{F}$ is full if the induced function $\mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(A), \mathcal{F}(B))$ is surjective for every pair of objects A and B in $\mathcal{C}$.

3. $\mathcal{F}$ is essentially surjective if for any object Y in $\mathcal{D}$, there is an object X in $\mathcal{C}$ such that Y isomorphic to $\mathcal{F}(X)$.

$\mathcal{F}$ is said to be an equivalence of categories if it is full, faithful and essentially surjective.

Definition. Given a locally small category $\mathcal{C}$ and a object A, we define the covariant functor $\mathrm{Hom}(A, \cdot) : \mathcal{C} \to \mathbf{Sets}$ and contavariant functor $\mathrm{Hom}(\cdot, A) : \mathcal{C} \to \mathbf{Sets}$. In $\mathrm{Hom}(A, \cdot)$ an object B is taken to $\mathrm{Hom}(A, B)$ and any morphism is $f : X \to Y$ is taken to the morphism (function) $(- \circ f) : \mathrm{Hom}(A, X) \to \mathrm{Hom}(A, Y)$ which for any morphism $g \in \mathrm{Hom}(A, X)$ takes it to $g \circ f$. $\mathrm{Hom}(\cdot, A)$ is defined similarly except that it is contravariant.

Definition. Given an object S in a category $\mathcal{C}$, we define the slice category over S to be a category $\mathcal{C}_S$ where objects consists of an object A and a morphism $s_A : A \to S$ in $\mathcal{C}$ and a morphism between objects $s_A : A \to S$ and $s_B : B \to S$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "s_A"'] & & B \arrow[dl, "s_B"] \\ & S & \end{tikzcd}$$

commutes.

Definition. Given a diagram

$$\begin{tikzcd} & A \arrow[d] \\ B \arrow[r] & S \end{tikzcd}$$

we define the fiber product (or pullback) to be an object $A \times_S B$ and morphisms $p_1 : A \times_S B \to A$ and $p_2 : A \times_S B \to B$ such that the diagram

$$\begin{tikzcd} A \times_S B \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d] \\ B \arrow[r] & S \end{tikzcd}$$

commutes, and if given any other object X and morphisms $\phi_1 : X \to A$ and $\phi_2 : X \to B$ which commutes with the original diagram, then there exists a unique morphism $m : X \to A \times_S B$ such that the diagram

$$\begin{tikzcd} X \arrow[dr, dotted, "m"] \arrow[drr, bend left, "\phi_1"] \arrow[ddr, bend right, "\phi_2"'] & & \\ & A \times_S B \arrow[r, "p_1"'] \arrow[d, "p_2"] & A \arrow[d] \\ & B \arrow[r] & S \end{tikzcd}$$

commutes.

Proposition 1.50. *The fiber product if exists is unique up to unique isomorphism.*

Definition. Given a morphism $f : A \to B$, it is called a $\mathsf{monomorphism}$ if for any pair of morphisms $h_1, h_2 : X \to A$, if $f \circ h_1 = f \circ h_2$, then $h_1 = h_2$. f is called an $\mathsf{epimorphism}$ if for any pair of morphisms $g_1, g_2 : B \to Y$, if $g_1 \circ f = g_2 \circ f$, then $g_1 = g_2$. This is analogous to the idea of injectivity and surjectivity respectively.

Definition. Given a morphism $0_{AB} : A \to B$, we say that it is a $\mathsf{zero\ morphism}$ if for any object X and Y and morphisms $g_1, g_2 : X \to A$ and $h_1, h_2 : B \to Y$, $f \circ g_1 = f \circ g_2$ and $h_1 \circ f = h_2 \circ f$. Given a morphism $f : A \to B$, we say that $\ker : K \to A$ is the kernel of f if $f \circ \ker = 0_{KY}$ and given any object K' and morphisms $k' : K' \to A$ such that $f \circ k' = 0_{K'B}$, then, there exists a unique morphism $m : K' \to K$ such that the diagram

$$\begin{tikzcd}
K \arrow[r, "\ker"] & A \arrow[r, "f"] & B \\
K' \arrow[u, dashed, "m"] \arrow[ur, "k'"'] & &
\end{tikzcd}$$

commutes. A morphism $\mathsf{coker} : B \to C$ is know as the $\mathsf{cokernel}$ if $\mathsf{coker} \circ f = 0_{AC}$ and for any morphism $c : B \to X$ such that $c' \circ f = 0_{AX}$ there is a unique morphism $n : C \to X$ such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r, "\mathsf{coker}"] \arrow[dr, "c"'] & C \arrow[d, dashed, "n"] \\
& & K'
\end{tikzcd}$$

Definition. Given a collection of objects $\{A_i\}_{i \in I}$, we define the $\mathsf{direct\ sum}$ (or $\mathsf{coproduct}$) denoted by $\bigoplus_{i \in I} A_i$ to be an object A along with maps $f_i : A_i \to A$ such that for any object B, if there exists morphisms $g_i : A_i \to B$, then there exists a unique morphism $f : A \to B$ such that $f \circ f_i = g_i$ for each i.

Definition. A partially ordered set I is said to be a $\mathsf{directed\ set}$ if for any $i, j \in I$, there exists a $k \in I$ such that $i \leq k$ and $j \leq k$.
Given a category $\mathcal{C}$ and a directed set I, a $\mathsf{directed\ system}$ indexed by I, is a collection of objects $\{A_i\}_{i \in I}$ and for any $i, j \in I$, if $i \leq j$, there is a morphism $\psi_{ij} : A_i \to A_j$ such that $\psi_{ii} = \mathrm{id}_{A_i}$ and $\psi_{jk} \circ \psi_{ij} = \psi_{ik}$ for some $i \leq j \leq k$ in I. A $\mathsf{direct\ limit}$ (or $\mathsf{colimit}$) of the the directed system (denoted by $\varinjlim A_i$) is an object A and morphisms $\psi_i : A_i \to A$ such that $\psi_j \circ \psi_{ij} = \psi_i$ whenever $i \leq j$ with the additional property that for any object B and morphisms $\phi_i : A_i \to B$ such that $\phi_j \circ \psi_{ij} = \phi_i$ whenever $i \leq j$, there is a unique morphism $f : A \to B$ such that $f \circ \psi_i = \phi_i$.

Proposition 1.51. *The direct limit of any directed system (if exists) is unique up to isomorphism.*

1.4 Homological Algebra

Definition. An $\mathsf{abelian\ category}$ is a category $\mathcal{C}$ such that: for any two objects A, B, $\mathrm{Hom}(A, B)$ is an abelian group with the composition law being linear; finite direct sums exists; every morphism has a kernel and a cokernel; every monomorphism is the kernel of its cokernel, every epimorphism is the cokernel of its kernel; and every morphism can be factored into an epimorphism followed by a monomorphism

Proposition 1.52. 5-Lemma. *Consider the following commutative diagram in a abelian category*

$$\begin{tikzcd} A_1 \arrow[r] \arrow[d, "a"'] & B_1 \arrow[r] \arrow[d, "b"'] & C_1 \arrow[r] \arrow[d, "c"'] & D_1 \arrow[r] \arrow[d, "d"'] & E_1 \arrow[d, "e"'] \\ A_2 \arrow[r] & B_2 \arrow[r] & C_2 \arrow[r] & D_2 \arrow[r] & E_2 \end{tikzcd}$$

such that both rows are exact. Moreover, suppose b *and* d *are isomorphisms,* a *is an epimorphism and* e *is a monomorphism. Then,* c *is an isomorphism.*

Definition. Given a morphism $f : A \to B$, the image of the morphism is an object I and a monomorphism $i : I \to B$ such that there exists an morphism $e : A \to I$ such that the following diagram

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "e"'] & B \\ & I \arrow[u, "i"'] \end{tikzcd}$$

commutes. And given any object I', monomorphism $i' : I' \to B$ and a morphism $e' : A \to I'$ such that $f = i' \circ e'$, then there exists a unique morphism $m : I \to I'$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr] & B \\ I' \arrow[ur] & I \arrow[u] \arrow[l, dashed, "m"'] \end{tikzcd}$$

Definition. A sequence of arrows and morphisms $\cdots \to A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \to \cdots$ is said to be **exact** at A_i if the kernel of f_i is the image of f_{i-1}. The sequence if exact if it is exact at each of its objects. An exact sequence of the form $0 \to A' \to A \to A'' \to 0$ is said to be a **short exact sequence**.

Definition. We say that a short exact sequence $0 \to A' \xrightarrow{f} A \xrightarrow{g} A" \to 0$ **splits** if any of the following equivalent statements are true:

1. $A \cong A' \bigoplus A"$ and the morphism $A' \xrightarrow{f} A$ and $A \xrightarrow{g} A"$ in the exact sequence are the natural maps.
2. There exists a morphism $f' : A \to A'$ such that $f' \circ f = id_{A'}$.
3. There exists a morphism $g" : A" \to A$ such that $g \circ g" = id_{A"}$

Definition. A covariant functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between abelian categories is said to be **additive** if for any two objects A, A' in $\mathcal{C}$, the induced map $Hom_{\mathcal{C}}(A, A') \to Hom_{\mathcal{D}}(\mathcal{F}A, \mathcal{F}A')$ is a group homomorphism. One can define additive for contravariant functors in a similar manner.

Definition. A covariant functor $F : \mathcal{C} \to \mathcal{D}$ between abelian categories is said to be **left exact** if $\mathcal{F}$ is additive and for any short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{C}$, the sequence $0 \to FA' \to FA \to FA''$ is exact.
A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ between abelian categories is said to be **left exact** if $\mathcal{F}$ is additive and for any short exact sequence $0 \to A' \to A \to A'' \to 0$ in $\mathcal{C}$, the sequence $0 \to FA'' \to FA \to FA'$ is exact.
Similarly, one defines a **right exact** functor. A functor is said to be **exact** if it is both left and right exact.

Proposition 1.53. *Given a abelian category* $\mathcal{C}$ *and an object* A*, the contravariant functor* $\mathsf{Hom}(\cdot, A)$ *and the covariant functor* $\mathsf{Hom}(A, \cdot)$ *are both left exact (Since it is an abelian category we can consider these functors from* $\mathcal{C}$ *to* ***Ab*** *rather than* ***Sets****).*

Definition. Given an abelian category $\mathcal{C}$ we define an object I to be **injective** if the functor $\mathsf{Hom}(\cdot, I)$ is exact. The category is said to have **enough injectives** if for any object A, there exists an injective object I and a monomorphism $i : A \to I$. Similarly, one says that an object P is projective if $\mathsf{Hom}(P, \cdot)$ is exact.

Proposition 1.54. *Suppose* $\mathcal{C}$ *is an abelian category with enough injectives. Then for any object* A*, there exists an exact sequence* $0 \to A \to I^0 \to I^1 \to \cdots \to I^i \to \cdots$*, where each of these objects* I^i *is injective. Such an exact sequence is called an* **injective resolution** *of* A*.*

Proposition 1.55. *Let* $0 \to A' \to A \to A" \to 0$ *be a short exact sequence and suppose* A' *is injective. Then, the sequence splits and* $A"$ *is also injective.*

Definition. Given an A-module M, and a **projective resolution**, $\cdots \to L_2 \to L_1 \to L_0 \to M \to 0$, we define the length of the projective resolution to be the n such that $L_n \neq 0$ and $L_m = 0$ for all $m > n$. We define the **projective dimension** of M, $\mathsf{pd}_A M$, to be the least length of a projective resolution.

Proposition 1.56. *Let* A *be a regular local ring do dimension* n*, and let* M *be a finitely generated* A*-module. Then, we have* $\mathsf{pd}\ M + \mathsf{depth}\ M = n$

Definition. Given an abelian category $\mathcal{C}$, we define a complex $A^\bullet$ to be a collection of objects $\{A^i\}_{i \in \mathbb{Z}}$ and morphisms $d^i : A^i \to A^{i+1}$ such that $d^{i+1} \circ d^i = 0$ for all $i \in \mathbb{Z}$. As an abuse of notation we write $d^2 = 0$. The i^{th} cohomology group of the complex $A^\bullet$, denoted by $h^i(A^\bullet) = {}^{\ker(d^i)}/{}_{\mathrm{im}(d^{i-1})}$.

Definition. Given two complexes $A^\bullet$ and $B^\bullet$, a morphism $f : A^\bullet \to B^\bullet$ is a collection of morphism $f^i : A^i \to B^i$ for each $i \in \mathbb{Z}$ such that $d^{i+1} \circ f^i = f^{i+1} \circ d^i$.

Proposition 1.57. *Given a short exact sequence of chain complexes* $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$*, there exists a collection of morphisms* $h^i(C^\bullet) \xrightarrow{\delta^i} h^{i+1}(A^\bullet)$ *for each* $i \in \mathbb{Z}$ *such that there exists a long exact sequence* $\cdots \to h^i(A^\bullet) \to h^i(B^\bullet) \to h^i(C^\bullet) \xrightarrow{\delta^i} h^{i+1}(A^\bullet) \to \cdots$

We say that two morphisms $f, g : \mathbb{A}^\bullet \to B^\bullet$ to be homotopic if if there exists a collection of morphisms $k^i : A^i \to B^{i-1}$ such that $f - g = dk + kd$. Homotopic morphism induce the same morphism on cohomology groups.

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories and suppose $\mathcal{C}$ has enough injectives. Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a covariant left exact functor. For each object A, fix an injective resolution $0 \to A \to I_A^0 \to I_A^1 \to \cdots \to I_A^i \to \cdots$. Then, for each $i \geq 0$, we define the **right derived functors** $R^i\mathcal{F} : \mathcal{C} \to \mathcal{D}$ such that $R^i\mathcal{F}(A) = h^i(\mathcal{F}(I_A^\bullet))$, where $\mathcal{F}(I_A^\bullet)$ is the chain $0 \to \mathcal{F}(I_A^0) \to \mathcal{F}(I_A^1) \to \cdots \to \mathcal{F}(I_A^i) \to \cdots$.

Proposition 1.58. *Let* $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ *be a covariant left exact functor of abelian categories. Suppose* $\mathcal{C}$ *has enough injectives. Then,*

1. *For all* $i \geq 0$*,* $R^i\mathcal{F}$ *is an additive functor. Furthermore, it is independent of the choice of resolution;*

2. *$\mathcal{F} \cong R^0\mathcal{F}$;*

3. *For any short exact sequence $0 \to A' \to A \to A" \to 0$, then there exists morphisms $\delta^i : R^i\mathcal{F}(A") \to R^{i+1}\mathcal{F}(A')$ for all $i \geq 0$ such that there exists a long exact sequence $0 \to R^0\mathcal{F}(A') \to R^0\mathcal{F}(A) \to R^0\mathcal{F}(A") \xrightarrow{\delta^0} R^1\mathcal{F}(A') \to \cdots \xrightarrow{\delta^{i-1}} R^i\mathcal{F}(A') \to R^i\mathcal{F}(A) \to R^i\mathcal{F}(A") \xrightarrow{\delta^i} R^{i+1}\mathcal{F}(A') \to \cdots$;*

4. *Given an morphism of the exact sequence in (3) to the exact sequence $0 \to B' \to B \to B" \to 0$, the following diagram*

$$\begin{tikzcd} R^i\mathcal{F}(A") \arrow[r, "\delta^i"] \arrow[d] & R^{i+1}\mathcal{F}(A') \arrow[d] \\ R^i\mathcal{F}(B") \arrow[r, "\delta^i"] & R^{i+1}\mathcal{F}(B') \end{tikzcd}$$

   commutes;

5. *For any injective object I in $\mathcal{C}$, and for any $i > 0$, $R^i\mathcal{F}(I) = 0$.*

Definition. With the same set up as the above proposition, we define an object J in $\mathcal{C}$ to be $\mathsf{acyclic}$ if $R^i\mathcal{F}(J) = 0$ for all $i > 0$.

Proposition 1.59. *With the same set up as the above proposition, let $0 \to A \to J^0 \to J^1 \to \cdots \to J^i \to \cdots$ be an exact sequence such that each J^i is acyclic. Then, $R^i\mathcal{F}(A) \cong h^i(\mathcal{F}(J^\bullet))$ for all $i \geq 0$. Thus, we can replace injective resolution with an acyclic resolution to calculate the derived functor.*

Definition. Let M be a fixed A-module. Then, we define the functor $\mathrm{Ext}^i_A(M, \cdot)$ to be the right derived functor of the left exact functor $\mathrm{Hom}_A(M, \cdot)$ from the category of A-modules to the category of abelian groups.

Proposition 1.60. *Let A be a ring, and M an A-module. Then, the following statements are equivalent:*

1. *$\mathrm{pd}_A\, M \leq n$;*

2. *$\mathrm{Ext}^i_A(M, N) = 0$ for all $i > n$ all A-modules N;*

3. *$\mathrm{Ext}^i_A(M, A) = 0$ for all $i > n$.*

Proposition 1.61. *If A is a noetherian ring and M and N are finitely generated A-modules, then $\mathrm{Ext}^i_A(M, N)$ is a finitely generate A-module.*

Definition. Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories. A covariant δ-$\mathsf{functor}$ from $\mathcal{C}$ to $\mathcal{D}$ is a collection of covariant functors $(T^i)_{i\geq 0}$ from $\mathcal{C}$ to $\mathcal{D}$ such that for any exact sequence $0 \to A' \to A \to A" \to 0$ in $\mathcal{C}$, there is a collection of functors $\delta^i : T^i(A") \to T^{i+1}(A')$ such that there exists an exact sequence $0 \to T^0(A') \to T^0(A) \to T^0(A") \xrightarrow{\delta^0} T^1(A') \to \cdots \to$

$T^i(A') \to T^i(A) \to T^i(A") \xrightarrow{\delta^i} T^{i+1}(A') \to \cdots$ and for any morphisms of the above short exact sequence to a short exact sequence $0 \to B' \to B \to B" \to 0$, the following diagram

$$\begin{tikzcd} T^i(A") \arrow[r, "\delta^i"] \arrow[d] & T^{i+1}(A') \arrow[d] \\ T^i(B") \arrow[r, "\delta^i"] & T^{i+1}(B') \end{tikzcd}$$

commutes. We can similarly define contravariat δ-$\mathsf{functors}$.

Definition. A δ-functor $(T^i) : \mathcal{C} \to \mathcal{D}$ is said to be $\mathsf{universal}$ if given any other δ-functor $(T'^i) : \mathcal{C} \to \mathcal{D}$, if there exists a morphism of functors $f^0 : T^0 \to T'^0$, then there exists unique morphisms $f^i : T^i \to T'^i$ for all $i > 0$ such that all the squares commute.

Definition. An additive functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is said to be $\mathsf{effaceable}$ if for each object A in $\mathcal{C}$, there exists an monomorphism $m : A \to I$ such that $\mathcal{F}(m) = 0$. $\mathcal{F}$ is said to be coeffaceable if for each object A, there exists an epimorphisms $e : P \to A$ such that $\mathcal{F}(e) = 0$.

Proposition 1.62. *Let $(T^i) : \mathcal{C} \to \mathcal{D}$ be a covariant δ-functor (contravariant δ-functor). If T^i is effaceable (coeffacable) for each $i > 0$, then (T^i) is universal.*

1.5 Affine and Projective Varieties

In this section we will always assume our field to be algebraically closed.

1.5.1 Affine Space

Given an field k, we define an $\mathsf{affine\ algebraic\ set}$ in k^n to be a the zeros set of a collection of polynomials $S \subseteq k[x_1, ..., x_n]$ (If the value of n and the field is understood or need not be specified, we shall denote $k[x_1, ..., x_n]$ by A)

Proposition 1.63. *1. $\emptyset$ and k^n are both algebraic sets.*

2. Arbitrary intersection of algebraic sets is an algebraic set.

3. Finite union of algebraic sets is an algebraic set.

Thus, it is clear that algebraic sets satisfy the properties of closed sets. And therefore we define the $\mathsf{Zariski\ Topology}$ on k^n by considering the closed sets to be algbraic sets. k^n with the zariski topology is called the affine n-space and denoted by $\mathbb{A}^n_k$ or simply $\mathbb{A}^n$

Proposition 1.64. *Let $S \subseteq A$. Let, $\mathfrak{a}$ be the ideal generated by S. Let $Z(S)$ denote the set of common zeros of the polynomials in S. Then:*

1. $Z(S) = Z(\mathfrak{a}) = Z(\mathrm{rad}(\mathfrak{a}))$.
2. *There exists polynomials $f_1, ..., f_k \in S$ such that $Z(S) = Z(f_1, ..., f_k)$, where $Z(f_1, ..., f_k)$ denotes $Z(\{f_1, ..., f_k\})$.*

If $W \subseteq \mathbb{A}^n$, then we denote by $I(W)$ the subset of A of all polynomials that vanish on W (one can check that this in fact is an ideal). We call $I(W)$ to be the ideal generated by W.

Proposition 1.65. *Hilbert's Nullstellensatz (Strong). For any ideal* $\mathfrak{a} \subseteq A$ *and polomial* f *which vanishes on* $Z(\mathfrak{a})$*, there exists an* $r \in \mathbb{N}$ *such that* $f^r \in \mathfrak{a}$*.*

Proposition 1.66. 1. *If* $T_1 \subseteq T_2 \subseteq k[x_1, ..., x_n]$*, then* $Z(T_2) \subseteq Z(T_1)$*.*

2. *If* $Y_1 \subseteq Y_2 \subseteq \mathbb{A}^n$*, then* $I(Y_2) \subseteq I(Y_1)$*.*

3. *For any subset* $W \subseteq \mathbb{A}^n$*,* $Z(I(W)) = \overline{W}$*.*

4. *For any ideal* $\mathfrak{a} \subseteq A$*,* $I(Z(\mathfrak{a})) = \mathrm{rad}(\mathfrak{a})$*.*

Corollary 1.67. *There is one-to-one inclusion-reversing correspondence between algebraic sets in* $\mathbb{A}^n$ *and radical ideal in* A*. Furthermore, an algebraic set is irreducible if and only if the ideal generated by it is prime.*

Definition. An algebraic set is called a $\mathsf{affine\ algebraic\ variety}$ if it is irreducible. An open subset of an affine variety is called a $\mathsf{quasi\text{-}affine\ varriety}$. Given an $f \in k[x_1, ..., x_n]$, we call $Z(f)$ to be a hypersurface.

Note that $\mathbb{A}^n$ is a noetherian topological space and therefore any closed set Y can be uniquely covered by varieties $Z_1, .., Z_k$ such that $Z_i \subseteq Z_j$ if and only if $i = j$.

Definition. Given an algebraic set W, we define the $\mathsf{affine\ coordinate\ ring}$ to be ${}^{A}/_{I(W)}$.

Proposition 1.68. *Given an algebraic set* Y*, its dimension as a topological space is the same of the dimension of its affine coordinate ring.*

Proposition 1.69. *The dimension of* $\mathbb{A}^n$ *is* n*.*

Proposition 1.70. *Given a quasi-affine variety* Y*,* $\dim Y = \dim \overline{Y}$*.*

Proposition 1.71. *A variety* Y *in* $\mathbb{A}^n$ *has dimension* $n - 1$ *if and only if it is the zero set* $Z(f)$ *of a single nonconstant irreducible polynomial in* $A = k[x_1, ..., x_n]$*.*

1.5.2 Projective Space

Given a field k. Define an equivalence realtion $\sim$ on $k^{n+1} \setminus \{0\}$ by taking $(a_0, ..., a_n) \sim (\lambda a_0, ..., \lambda a_n)$ where $\lambda \in k \setminus \{0\}$. Given a $(a_0, ..., a_n) \in k^n \setminus \{0\}$, we denote by $[a_0 : ... : a_n]$ its equivalence class in ${}^{k^{n+1}}/_{\sim}$.
Consider $k[x_0, ..., x_n]$ as a graded ring. Given a homogeneous polynomial f and a point $P = [a_0 : ... : a_n] \in {}^{k^{n+1}}/_{\sim}$, we say P is a zero of f if $f(a_0, ..., a_n) = 0$. One can show that this is well defined. We define a $\mathsf{projective\ algebraic\ set}$ to be the zero set of some collection of homogeneous polynomials. Given a homogeneous ideal $\mathfrak{a}$, we define the zero set of $\mathfrak{a}$ to be the zero set of all homogeneous elements of $\mathfrak{a}$.
Similar to the affine case, on can show that projective algebraic set satisfy the properties of closed sets.

Definition. Given a field k, we define the n^{th} $\mathsf{projective\ space}$ (denoted by $\mathbb{P}^n_k$, or $\mathbb{P}^n$ if it is not necessary to specify the field) to be the set ${}^{k^{n+1}}/_{\sim}$ as defined above and the closed sets being projective algebraic sets.

Definition. An irreducible projective algebraic set is called a $\mathsf{projective\ variety}$. An open subset of a projective variety is called a $\mathsf{quasi\text{-}projective\ variety}$.

For any subset $Y \subseteq \mathbb{P}^n$ we define the homogeneous ideal of Y (denoted by $I(Y)$) to be the ideal generated by the homogeneous polynomials f which are zeros of every point in Y. Like in the the case of affine case, we define the projective coordinte ring of Y to be $S(Y) = {}^{S}/_{I(Y)}$, where $S = k[x_0, ..., x_n]$.

We define a hyperplane to be the zero set of a single homogeneous polynomial of degree 1. For any $0 \leq i \leq n$, we define H_i to the hyperplane defined by x_i. We define $U_i = \mathbb{P}^n \setminus H_i$. Let $f_i : U_i \to \mathbb{A}^n$ such that $f([a_0 : ... : a_n]) = (\frac{a_0}{a_i}, ..., \frac{a_n}{a_i})$ where we omit $\frac{a_i}{a_i}$.

Proposition 1.72. *The function f_i defined above is a homeomorphism.*

Proposition 1.73. *The sets $U_0, ..., U_n$ cover $\mathbb{P}^n$. This is called the affine cover of $\mathbb{P}^n$.*

Corollary 1.74. *If Y is a projective (respectivel quasi-projective) variety, then Y is covered by the open sets $Y \cap U_i$, $i = 0, ..., n$, which are homeomorphic to affine (respectively quasi-affine) varieties via the map f_i defined above.*

1.5.3 Category of varieties

Definition. Given a quasi-affine variety Y, we say a function $f : Y \to k$ is regular at $P \in Y$ if there exists an open neighbourhood of P in Y and polynomials $p, g \in k[x_1, ..., x_n]$ such that $f|_U = {}^{p}/_{g}$. It is said to be regular on Y if it is regular at every point on Y.

Definition. Given a quasi-projective variety Y, we say that a function $f : Y \to k$ is regular at $P \in Y$ if there exists a neighbourhood $U \subseteq Y$ containing P and homogeneous polynomials $p, g \in k[x_0, ..., x_n]$ of the same degree such that $f|_U = {}^{p}/_{g}$

Definition. Given an algebraically closed field k, we define the category of varieties over k (denoted by $\mathrm{Var}(k)$). The objects of this category are all affine, quasi affine, projective and quasi projective varieties over k. We call an object of this category a $\mathtt{variety}$, A continuous function $f : X \to Y$ between two varieties is said to be a morphism if, given any regular function g over Y, then $g \circ f$ is a regular function over X.

Definition. For any variety Y, we define $\mathcal{O}(Y)$ to be the set of all regular functions on Y. For each $P \in Y$, we define the local ring at P denoted by $\mathcal{O}_{Y,P}$ (or $\mathcal{O}_P$) to be the ring of all germs of the form $< U, g >$, where $U \subseteq Y$ is an open set containing P and $g \in \mathcal{O}_P$ such that two such germs $< U, g >$ and $< V, h >$ are said to be equal if $h|_U = g|_V$.

Proposition 1.75. *For any variety Y over k and $P \in Y$, $\mathcal{O}_P$ is a local ring. If m is its unique maximal ideal then ${}^{\mathcal{O}_P}/_{m} \cong k$*

Definition. For any variety Y, we define the function field of Y, denoted by $K(Y)$, to be the set of all germs of the form $< U, g >$ under the same equivalence relation stated above.

Proposition 1.76. *For any variety Y, $K(Y)$ is a field and for any point $P \in Y$, there is a sequence of injective maps $\mathcal{O}(Y) \to \mathcal{O}_{P,Y} \to K(Y)$.*

Proposition 1.77. *For any two varieties Y and Z, if $Y \cong Z$, then the coresponding rings defined above are isomorphic.*

Proposition 1.78. *Let $Y \subseteq \mathbb{A}^n$ be an affine variety with affine coordinate ring $A(Y)$. Then:*

1. *$\mathcal{O}(Y) \cong A(Y)$;*

2. *for each point* $P \in Y$*, let* $\mathfrak{m}_P \subseteq A(Y)$ *be the ideal of functions vanishing at* P*. Then* $P \mapsto \mathfrak{m}_P$ *gives a one to one correspondence between the point of* Y *and maximal ideals of* $A(Y)$*;*

3. *for each* P, $\mathcal{O}_P \cong A(Y)_{\mathfrak{m}_P}$*, and* $\dim \mathcal{O}_P = \dim Y$*;*

4. $K(Y)$ *is isomorphic to the quotient field of* $A(Y)$ *and hence* $K(Y)$ *is a finitely generated field extension of* k*, of transcedence degree* $= \dim Y$*.*

Proposition 1.79. *The function* $f_i : U_i \to \mathbb{A}^n$ *defined in the previous section is in fact an isomorphism in the category of varieties.*

Proposition 1.80. *Let* $Y \subseteq \mathbb{P}^n$ *be a projective variety with homogeneous coordinate ring* $S(Y)$*. Then:*

1. $\mathcal{O}(Y) = k$*;*

2. *for each point* $P \in Y$*, let* $\mathfrak{m}_P \subseteq S(Y)$ *be the ideal generated by the homogeneous functions vanishing at* P*, then* $\mathcal{O}_P \cong S(Y)_{(\mathfrak{m}_P)}$*;*

3. $K(Y) \cong S(Y)_{((0))}$.

Chapter 2

Schemes

In this chapter all results which can easily be found and are well explain in Harstshorne shall be cited.

2.1 Sheaves

Definition. Given a topological space X, we define a $\mathtt{presheaf}$ of abelian groups to be a contravariant functor $\mathcal{F} : \mathbf{Top(X)} \to \mathbf{Ab}$ with $F(\emptyset) = 0$

Remark. Firstly, If $V \subseteq U$ and given an element $s \in \mathcal{F}(U)$, we denote the image of s in $\mathcal{F}(V)$ by $s|_V$. Secondly, we could also look at presheaf of rings or sets or any other suitable category with the only difference being that the functor would then be to that category.

Definition. Given a topological space we define a $\mathtt{sheaf}$ to be a presheaf $\mathcal{F}$ such that:

1. For any open set U and open covering $\{U_i\}$ of U, if $s \in \mathcal{F}(U)$ such that $s|_{U_i} = 0$, then $s = 0$.

2. For any open set U and open covering $\{U_i\}$ of U, if $s_i \in \mathcal{F}(U_i)$ such that for any i, j, $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$, then there exists an $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.

Elements of $\mathcal{F}(U)$ are called $\mathtt{sections}$

Example. For any topological space X and given any abelian group A, we define the $\mathtt{constant\ sheaf}$ $\mathcal{A}$ on X. First we give A the discrete topology (i.e. every subset of A is an open set) and for any open set $U \subseteq X$, we define $\mathcal{A}(U)$ to be the set of continuous functions from U to A and for any pair of open sets $V \subseteq U$, we define $\mathcal{F}(U) \to \mathcal{F}(V)$ to be the natural restriction map.

Definition. Given a presheaf $\mathcal{F}$ on X, a $\mathtt{subpresheaf}$ is a presheaf $\mathcal{F}'$ on X such that for each open set $U \subseteq X$, $\mathcal{F}'(U) \subseteq \mathcal{F}(U)$. We define a subsheaf of a sheaf similarly.

Definition. Given presheaves $\mathcal{F}$ and $\mathcal{G}$ on a topological space X, we define a morphism of presheaves $\phi : \mathcal{F} \to \mathcal{G}$ as a collection of morphisms $\phi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ for each open subet $U \subseteq X$ such that for any pair of open sets $V \subseteq U$, the following diagram

$$\begin{tikzcd} \mathcal{F}(V) \arrow[r, "\phi(V)"] & \mathcal{G}(V) \\ \mathcal{F}(U) \arrow[u] \arrow[r, "\phi(U)"] & \mathcal{G}(U) \arrow[u] \end{tikzcd}$$

commutes. We give the same definition for morphisms of sheaves. One can check that this satisfies all the properties a morphism in a category and thus we can define the category of presheaves and also the category of sheaves of abelian groups over X.

Definition. Given a presheaf $\mathcal{F}$ on X, and a point $p \in X$, we define the stalf $\mathcal{F}_{X,p}$ (or $\mathcal{F}_p$) at p to be the direct limit $\varinjlim_{p \in U} \mathcal{F}(U)$, where U denotes opens sets that contain p. For any section $s \in \mathcal{F}(U)$ and $p \in U$ we denote the image of s in $\mathcal{F}_p$ by s_p.

Proposition 2.1. *A morphism $\phi : \mathcal{F} \to \mathcal{G}$ of sheaves on X is an isomorphism if and only if the natural map $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ is an isomorophism for ever $p \in X$.*

Proof. [3, page 63] □

Proposition 2.2. *Given any presheaf $\mathcal{F}$, there exists a sheaf $\mathcal{F}^+$ and a morphism $\phi : \mathcal{F} \to \mathcal{F}^+$ such that for any sheaf $\mathcal{G}$ and morphism $\psi : \mathcal{F} \to \mathcal{G}$, there exists a unique morphism $f : \mathcal{F}^+ \to \mathcal{G}$ such that $f \circ \phi = \psi$.*

Proof. [3, page 64] □

Proposition 2.3. *For any presheaf $\mathcal{F}$ on X, and $p \in X$, $\mathcal{F}_p \cong \mathcal{F}^+_p$, where $\mathcal{F}^+$ is as defined above.*

Proof. For any open set U containing p, there is a natural map from $\phi_p(U) : \mathcal{F}^+(U) \to \mathcal{F}_p$. Suppose G is a group and there is a collection of homomorphisms $\psi(U) : \mathcal{F}^+(U) \to G$ such that all the diagrams commute. Now, we know that there are maps $\mathcal{F}(U) \to \mathcal{F}^+(U)$. Thus, there are maps $\mathcal{F}(U) \to G$ for every U containing p and all neccessary diagrams commuting. Therefore, there is a natural map $\mathcal{F}_p \to G$. □

Definition. The sheaf $\mathcal{F}^+$ defined above is the called the $\mathsf{sheaf\ associated\ to\ the\ presheaf}$ $\mathcal{F}$

Proposition 2.4. *Given a sheaf morphism $\phi : \mathcal{F} \to \mathcal{G}$ of sheaves of abelian groups on X, the presheaf $U \to \ker(\phi(U))$ is a sheaf.*

Proof. Let U be an open set and $\{U_i\}$ an open cover of U. Now, since $\ker \phi$ is a subpresheaf of a sheaf, it inherits the property that if $t \in (\ker \phi)(U)$ such that $t|_{U_i} = 0$ for all i, then $t = 0$. Thus, it is adequate to show that if $t_i \in (\ker \phi)(U_i)$ such that $t_i|_{U_i \cap U_j} = t_j|_{U_i \cap U_j}$ for all i and j, then there exists a $t \in (\ker \phi)(U)$ such that $t|_{U_i} = t_i$. Now, each $t_i \in \mathcal{F}(U_i)$, so it is clear that there exists a $t \in \mathcal{F}(U)$, such that $t|_{U_i} = t_i$. let $\phi(U)(t) = s \in \mathcal{G}(U)$. Now, $s|_{U_i} = \phi(U_i)(t|_{U_i}) = \phi(U_i)(t_i) = 0$. Therefore, $s = 0$. Thus, $t \in (\ker \phi)(U)$ □

Definition. Given a sheaf morphism $\phi : \mathcal{F} \to \mathcal{G}$ of sheaves of abelian groups on X, we define the kernel ($\ker\phi$) to be the sheaf such that $U \to \ker(\phi(U))$. We define the image ($\mathrm{im}\ \phi$) to be the sheaf associated to the presheaf $U \to \mathrm{img}(\phi(U))$ and we define the $\mathsf{cokernel}$ ($\mathrm{coker}\ \phi$) to be the sheaf associated to the presheaf $U \to \mathrm{coker}(\phi(U))$.

Definition. Given a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$, ϕ is said to be injective if $\ker \phi$ is the zero sheaf and ϕ is said to be surjective if $\mathrm{im}\ \phi = \mathcal{G}$

Proposition 2.5. *Given a morphism of sheaves $\phi : \mathcal{F} \to \mathcal{G}$ on X, ϕ is injective (or surjective) if and only if the induces morphism $\phi_p : \mathcal{F}_p \to \mathcal{G}_p$ is injective (or surjective) for all $p \in X$*

Proof. A simple calculation will show that $(\mathsf{ker}\ \phi)_p \cong \mathsf{ker}\ \phi_p$ and $(\mathsf{img}\ \phi)_p \cong \mathsf{img}\ \phi_p$ for any $p \in X$. Now, ϕ is injective if and only if $\mathsf{ker}\ \phi = 0$. Thus, for any point $p \in X$, $\mathsf{ker}\ \phi_p \cong (\mathsf{ker}\ \phi)_p = 0$ and so ϕ_p is injective if and only if ϕ is injective. Similarly ϕ is surjective if and only if $\mathsf{img}\ \phi = \mathcal{G}$. Thus, for any $p \in X$, $\mathsf{img}\ \phi_p \cong (\mathsf{img}\ \phi)_p \cong \mathcal{G}_p$ and therefore ϕ_p is surjective if and only if ϕ is surjective. □

Definition. Given a sequene of sheaves and sheaf morphisms $\cdots \to \mathcal{F}_{i-1} \xrightarrow{\phi_{i-1}} \mathcal{F}_i \xrightarrow{\phi_i} \mathcal{F}_{i+1} \to \cdots$ we say that it is exact at $\mathcal{F}_i$ if $\mathsf{ker}\phi_i = \mathsf{im}\phi_{i-1}$

Definition. For any topological space X, we define the functor $\Gamma(X, \cdot) : \mathsf{Sh}(X) \to \mathbf{Ab}$ (where $\mathsf{Sh}(X)$ are sheaves of abelian groups on X), such that for any sheaf $\mathcal{F}$, $\Gamma(X, \mathcal{F})$ is the group of global sections. This functor could have a different source and target if the sheves are defined over other structures like sets, rings, etc.

Proposition 2.6. $\Gamma(X, \cdot)$ *is left exact.*

Proof. This is a direct consequence of the fact that the presheaf $\mathsf{ker}\ \phi$ is a sheaf. □

Definition. Let $f : X \to Y$ be a continuous function of topological spaces. Suppose $\mathcal{F}$ is a sheaf on X. We define the sheaf $f_*\mathcal{F}$ on Y such that for any open set $U \subseteq Y$, $(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$. Let $\mathcal{G}$ be a shead of on Y. We define the sheaf $f^{-1}\mathcal{G}$ on X such that for any open subset $V \subseteq X$, $(f^{-1}\mathcal{G})(V) = \varinjlim_{U \subseteq f(V)} \mathcal{G}(U)$.
If $Z \subseteq X$, then we define $\mathcal{F}|_Z$ to be the sheaf $i^{-1}\mathcal{F}$ on Z, where $i : Z \to X$ is the inclusion map.

Definition. A sheaf $\mathcal{F}$ on X is said to be $\mathsf{flasque}$ if for any open seubsets $U \subseteq V \subseteq X$, the map $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective.

Proposition 2.7. *A sheaf* $\mathcal{F}$ *on* X *is* $\mathsf{flasque}$ *if and only if for any open subset* $U \subseteq X$*, the map* $\mathcal{F}(X) \to \mathcal{F}(U)$ *is surjetive.*

Proof. One way is clear. If $\mathcal{F}$ is flasque, then $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjetive for every open set U. Now, suppose $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjetive for every open set U. Let $V \subseteq W$ be open subsets. The, the map $\mathcal{F}(X) \to \mathcal{F}(V)$ factors through $\mathcal{F}(W)$. Thus, the map $\mathcal{F}(W) \to \mathcal{F}(V)$ is also surjective. □

Proposition 2.8. *Let* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *be an exact sequence of sheaves on* X*. If* $\mathcal{F}'$ *is flasque, then* $0 \to \Gamma(X, \mathcal{F}') \to \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}'') \to 0$ *is exact.*

Proof. Since $\Gamma(X, \cdot)$ is left exact, it is enough to show that $\Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{F}")$ is surjective. Let $s \in \Gamma(X, \mathcal{F}")$ be a global section. Since the morphism $\phi : \mathcal{F} \to \mathcal{F}"$ is surjective, one can easily see that there exists an open cover $\{U_i\}_{i \in I}$ of X and $s_i \in \mathcal{F}(U_i)$ such that $\phi_{U_i}(s_i) = s|_{U_i}$. Now, define a partially ordered set as follows: Let the elements be (J, f) where $J \subseteq I$ and $f \in \mathcal{F}(\bigcup_{i \in J} U_i)$ such that $\phi_{\bigcup_{i \in J} U_i}(f) = s|_{\bigcup_{i \in J} U_i}$ and $(J_1, g) \leq (J_2, f)$ when $J_1 \subseteq J_2$ and $f|_{\bigcup_{i \in J_1} U_i} = g$. Now, this set is non empty and by the definition of sheaves has a maximal element. Let (I_0, z) be that maximal elements. Let us assume that $I_0 \neq I$ and fix $i \in I \backslash I_0$. For convenience we shal denote $V = \bigcup_{j \in I_0} U_j$. Now, based on our choice of the cover, there exists an $f \in \mathcal{F}(U_i)$ such that $\phi_{U_i}(f) = s|_{U_i}$. Thus, $z|_{U_i \cap V} - f|_{U_i \cap V} \in \mathsf{ker}\ \phi_{U_i \cap V}$. Now, $\mathsf{ker}\ \phi = \mathcal{F}'$ is flasque. Thus, there exists a $t' \in \mathsf{ker}\ \phi_{U_i}$ such that $t'|_{U_i \cap V} = z|_{U_i \cap V} - f|_{U_i \cap V}$. Let $t = f + t'$. Now, $t|_{U_i \cap V} = f|_{U_i \cap V} + t'|_{U_i \cap V} = f|_{U_i \cap V} + z|_{U_i \cap V} - f|_{U_i \cap V} = z|_{U_i \cap V}$. Therefore, there exists a section $x \in \mathcal{F}(U_i \cup V)$ such that $x|_V = z$ and $x|_{U_i} = t$ and therefore, $\phi_{U_i \cup V}(x) = s|_{U_i \cup V}$. Thus, we have $(I_0, z) < (I_0 \cup \{i\}, x)$. This is a contradiction. Thus, $I_0 = I$ and therefore $z \in \mathcal{F}(\bigcup_{j \in I} U_j) = \mathcal{F}(X)$ is a desired preimage of s. □

Definition. Extending a sheaf by zero. Let X be a topological space (i in both cases represent the respective inclusion maps).

1. Given an closed sub subset $Z \subseteq X$ and a sheaf $\mathcal{F}$ on Z, we call the sheaf $i_*\mathcal{F}$ to the sheaf obtained by extending $\mathcal{F}$ by zero outside Z.

2. Let $U \subseteq X$ be an open subset and $\mathcal{F}$ be a sheaf of U. We define the sheaf $i_!\mathcal{F}$ on X by taking $(i_!\mathcal{F})(V) = \mathcal{F}(V)$ if $V \subseteq U$ and zero for every other open set. We call the sheaf $i_!\mathcal{F}$ to be the sheaf obtained by extending $\mathcal{F}$ by zero outside U.

Proposition 2.9. *Let X be a topological space and $Y \subseteq X$ be an closed (or open) subset and $\mathcal{F}$ a sheaf on Y. Then, for any $p \in X$, $i_*\mathcal{F}_p = 0$ (or $i_!\mathcal{F} = 0$) whenever $p \notin Y$ and $i_*\mathcal{F}_p = \mathcal{F}_p$ (or $i_!\mathcal{F}_p = \mathcal{F}_p$) whenever $p \in Y$.*

Proof. We shall prove the case when Y is closed, since it will be required later. But, the open case has a similar proof. Let $p \in X$ be a point. First, suppose $p \in Y$. Now, $(j_*\mathcal{F})_p = \varinjlim_{p\in U}(j_*\mathcal{F})(U) = \varinjlim_{p\in U}\mathcal{F}(j^{-1}(U))$. The open sets of the form $j^{-1}(U)$ are precisely all open sets in Y containing p. Thus, $\varinjlim_{p\in U}\mathcal{F}(j^{-1}(U)) = \mathcal{F}_p$. Therefore, $(j_*\mathcal{F})_p = \mathcal{F}_p$. Now, suppose $p \notin Y$. Let s be a stalk of $j_*\mathcal{F}$ at p. Thus, there exists an open set $U \subseteq X$ containing p with a section say t whose restriction to p is s. Now, the restriction of the section $t|_{U\cap Y^c}$ to p is also s. $(j_*\mathcal{F})(U \cap Y^c) = \mathcal{F}(j^{-1}(U \cap Y^c)) = 0$. Thus, $t|_{U\cap Y^c} = 0$. Therefore, $s = 0$. Thus, $(j_*\mathcal{F})_p = 0$. ☐

2.2 Schemes and Morphisms

Certain section are well explained in [2] and can be used for a good introduction to schemes. Other good references would be [6] and [7].

Let A be a ring. Denote by $\mathsf{Spec}\ A$ the set of prime ideal of A. For any ideal $\mathfrak{a} \subseteq A$, let $V(\mathfrak{a}) \subseteq \mathsf{Spec}\ A$ be the set of all prime ideal of A containing $\mathfrak{a}$.

Proposition 2.10. *Show that the sets of the form $V(\mathfrak{a})$ defined above satisfy the property of closed sets.*

Proof. [3, page 70] ☐

We can now consider $\mathsf{Spec}\ A$ as a topological space by taking the closed sets to be the sets of the form $V(\mathfrak{a})$ as defined above.

Proposition 2.11. *For any $f \in A$, we define $D(f) \subseteq \mathsf{Spec}\ A$ as the set of all prime ideal not containing f. Then, the sets of the form $D(f)$ form a basis for the topology on $\mathsf{Spec}\ A$*

Proof. Let U be an open subset of $\mathsf{Spec}\ A$. Now, $U = V(\mathfrak{a})^c$ for some ideal $\mathfrak{a} \subseteq A$. Then $U = \bigcup_{f\in\mathfrak{a}} D(f)$. ☐

Now, we shall define a sheaf $\mathcal{O}$ on $\mathsf{Spec}\ A$ as follows. Let $U \subseteq \mathsf{Spec}\ A$, then define $\mathcal{O}(U)$ as the set of all functions $s : U \to \coprod_{p\in U} A_p$ such that $s(p) \in A_p$ and for any prime $p \subseteq A$, there exists a subset $V \subseteq U$ containing p and $a, f \in A$ such for all $q \in V$, $f \notin q$ and $s(q) = {}^a/_f$

Definition. Give a ring A, the $\mathsf{spectrum}$ of A consists of the pair $\mathsf{Spec}\ A$ and the sheaf $\mathcal{O}$ defined on $\mathsf{Spec}\ A$ above.

Proposition 2.12. *Let* A *be a ring and* $(\mathsf{Spec}\ A, \mathcal{O})$ *its spectrum. Then:*

1. *For any* $p \in \mathsf{Spec}\ A$*,* $\mathcal{O}_p \cong A_p$*.*
2. *For any* $f \in A$*,* $\mathcal{O}(D(f)) \cong A_f$*. In particular* $\Gamma(\mathsf{Spec}\ A, \mathcal{O}) \cong A$

Proof. [3, page 71] □

Definition. A $\mathsf{ringed\ space}$ is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$. A morphism between ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, if a pair $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ where $f : X \to Y$ is a continuous function and $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves on Y. A ringed space $(X, \mathcal{O}_X)$ is a $\mathsf{locally\ ringed\ space}$ if for any $p \in X$, $\mathcal{O}_{X,p}$ is a local ring. Given two locally ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$, a morphism of ringed spaces $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is said to be a morphism of locally ringed spaces if the natural map that $f^\#_p : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is a local homomorphism.
Using the above definition one can define the category of locally ringed spaces.

Proposition 2.13.

1. *For a ring* A*,* $(\mathsf{Spec}\ A, \mathcal{O}_{\mathsf{Spec}\ A})$ *its spectrum is a locally ringed space.*
2. *If* A *and* B *are rings, then a homomorphism* $\phi : A \to B$ *induces natural morphism* $(f, f^\#) : (\mathsf{Spec}\ B, \mathcal{O}_{\mathsf{Spec}\ B}) \to (\mathsf{Spec}\ A, \mathcal{O}_{\mathsf{Spec}\ A})$ *of locally ringed spaces.*
3. *Given a morphisms* $(f, f^\#) : (\mathsf{Spec}\ B, \mathcal{O}_{\mathsf{Spec}\ B}) \to (\mathsf{Spec}\ A, \mathcal{O}_{\mathsf{Spec}\ A})$ *of locally ringed spaces, it induces a natural homomorphism* $\phi : A \to B$

Proof. [3, page 73] □

Definition. A locally ringed space $(X, \mathcal{O}_X)$ is called an $\mathsf{affine\ scheme}$ if there exists a ring A such that $(X, \mathcal{O}_X) \cong (\mathsf{Spec}\ A, \mathcal{O}_{\mathsf{Spec}\ A})$ is the category of locally ringed spaces. Further, a locally ringed space $(X, \mathcal{O}_X)$ is called a scheme if for each point $p \in X$, there exists an open set $U \subseteq X$ containing p such that $(U, \mathcal{O}_X|_U)$ is an affine scheme. For any scheme $(X, \mathcal{O}_X)$, the sheaf $\mathcal{O}_X$ is called its $\mathsf{structure\ sheaf}$. Sometimes, as a abuse of notation, we may refer to a scheme X if its structure sheaf is clear from the context and incase this could lead to a confusion with the topological space, it will be explicitely mentioned (In such cases, we sometime denote the topological space as $\mathsf{sp}(X)$). A morphism of schemes is a morphism of locally ringed spaces.

Definition. For any field k, we define the $\mathsf{affine}\ n\text{-}\mathsf{space}$ as a $\mathsf{Spec}\ k[x_1, ..., x_n]$ with its structure sheaf.

Definition. For any graded ring S, we define $\mathsf{Proj}\ S$. The elements of the space are all homogeneous prime ideals p that do not contain $S_+ = \bigoplus_{d>0} S_d$. Again, for any homogeneous ideal a, we define $V(a)$ to be all the elements of $\mathsf{Proj}\ S$ that contain a. One can check that these sets satisfy the properties of closed sets and thus defines a topology on $\mathsf{Proj}\ S$. Now we define a sheaf of rings $\mathcal{O}_{\mathsf{Proj}\ S}$. For each open set $U \subseteq \mathsf{Proj}\ S$, $\mathcal{O}_{\mathsf{Proj}\ S}(U)$ is the set of function $s : U \to \coprod_{p \in U} S_{(p)}$ such that for any point $p \in U$, $s(p) \in S_{(p)}$ and there exists a open subset $V \subseteq U$ containing p and homogeneous elements $a, f \in S$ of the same degree such that for all $q \in V$, $f \notin q$ and $s(q) = {}^a/_f$

Proposition 2.14. *Let S be a graded ring.*

1. *Let $\mathfrak{p} \in \mathsf{Proj}\ S$ be a prime ideal. Then, $\mathcal{O}_{X,\mathfrak{p}} \cong S_{(\mathfrak{p})}$ where $X = \mathsf{Proj}\ S$.*
2. *For any homogeneous $f \in S_+$, define $D(f)$ similar as above. Then, $D(f)$ is open in $\mathsf{Proj}\ S$. Furthermore, these open sets cover $\mathsf{Proj}\ S$, and for each such open set we have an isomorphism of locally ringed spaces $(D(f), \mathcal{O}_{\mathsf{Proj}\ S}|_{D(f)}) \cong \mathsf{Spec}\ S_{(f)}$*
3. *$\mathsf{Proj}\ S$ is a scheme.*

Proof. [3, page 76, 77] □

Definition. For any ring A, define the $\mathsf{projective}$ n-space over A to be $\mathbb{P}^n_A = \mathsf{Proj}\ S$, where $S = A[x_0, ..., x_n]$

Definition. For any scheme X and point $\mathfrak{p} \in X$, we define the $\mathsf{residue\ field}$ at $\mathfrak{p}$ to be $\mathcal{O}_{X,\mathfrak{p}}/\mathfrak{m}_\mathfrak{p}$ where $\mathfrak{m}_\mathfrak{p}$ is the maximal ideal of the local ring $\mathcal{O}_{X,\mathfrak{p}}$

Definition. Given a scheme S, we define the category of $\mathsf{schemes\ over}\ S$ to be the slice category over S in the category of schemes. We denote this category as $\mathsf{Sch}(S)$. For a ring A, we often write $\mathsf{Sch}(A)$ rather than $\mathsf{Sch}(\mathsf{Spec}\ A)$ as a abuse of notation and call it schemes over A.

Proposition 2.15. *The scheme $\mathsf{Spec}\ \mathbb{Z}$ is the final object in the category of schems.*

Proof. Let X be any scheme. Now, there exists a unique homomorphims $f(X) : \mathbb{Z} \to \Gamma(X, \mathcal{O}_X)$. For any affine open set $\mathsf{Spec}\ A = U \subseteq X$, the natural inclusion map of U into X induced a homomophims $f(U) : \mathbb{Z} \to A$. This induces a natural morphism of affine schemes $\mathsf{Spec}\ A \to \mathsf{Spec}\ \mathbb{Z}$. Joining these affine morphisms togethes induces a unique natural morphism $\phi : X \to \mathsf{Spec}\ \mathbb{Z}$ of schemes. Thus, $\mathsf{Spec}\ \mathbb{Z}$ is the final object in the category of schemes. □

Proposition 2.16. *Given an algebraically closed field k, there is natural fully faithful functor $t : \mathsf{Var}(k) \to \mathsf{Sch}(k)$. For any variety V, its topological space is homeomorphic to the set of closed points of $\mathsf{sp}(t(V))$ and its sheaf of regular functions is obtained by restricting the structure sheaf of $t(V)$ via this hoeomorphism.*

Proof. [3, page 78] □

Definition. Given a scheme X, we define the $\mathsf{dimension}$ of X as the dimension of its topological space.

Definition. A scheme is said to be $\mathsf{connected}$ or $\mathsf{irreducible}$ if its topological space is connected or irreducible respectively.

Definition. A scheme X is $\mathsf{reduced}$ if for every open set U, the ring $\mathcal{O}_X(U)$ has no non-zero nilpotent elements.

Proposition 2.17. *A scheme X is reduced if an only if for any point $\mathfrak{p} \in X$, $\mathcal{O}_{X,\mathfrak{p}}$ has no nilpotent elements.*

Proof. Suppose X is not reduced. Thus, there exists an open set U such that $\mathcal{O}_X(U)$ contains a non-zero nilpotent element say n. Now since $n \neq 0$, there exists a point $p \in U$ such that $n_p \neq 0$. Now, there exists a $k \in \mathbb{N}$ such that $n^k = 0$. Therefore, $(n_p)^k = 0$. Thus, there exists a point p such that $\mathcal{O}_{X,p}$ has a nilpotent element. Now, to prove the converse, we suppose that $p \in X$ is a point such that there there exists a non zero nilpotent element in $\mathcal{O}_{X,p}$ say s. Now, for some $k \in \mathbb{N}$, $s^k = 0$. Let U be an open set containing p on which s is defined. Since $s^k = 0$ in $\mathcal{O}_{X,p}$, there exists an open set $V \subseteq U$ containing p such that the image of s^k in the map $\mathcal{O}_X(U) \to \mathcal{O}_X(V)$ is zero. But, the image of s in V is not zero. Thus, $\mathcal{O}_X(V)$ has a non zero nilpotent element. □

Definition. A scheme X is said to be $\mathsf{integral}$ if for each open set U, $\mathcal{O}_X(U)$ is a domain.

Proposition 2.18. *A scheme is integral if and only if it is both irreducible and reduced.*

Proof. [3, page 82] □

Definition. A scheme is $\mathsf{locally\ noetherian}$ if it can be covered by open subsets $\mathrm{Spec}\ A_i$ where each A_i is noetherian. It is $\mathsf{noetherian}$ if it is locally noetherian and quasi compact.

Definition. Let X be a noetherian scheme. We call it $\mathsf{Cohen\text{-}Macaulay}$ is each of it's local rings are Cohen-Macaulay

Definition. A morphism $f : X \to Y$ of scheme is $\mathsf{locally\ finite\ type}$ if Y can be covered by some open subsets $\mathrm{Spec}\ B_i$ such that $f^{-1}(\mathrm{Spec}\ B_i)$ can be covered by some open subsets $\mathrm{Spec}\ A_{i,j}$ where each $A_{i,j}$ if a fintely generated B_i-$\mathsf{algebra}$. The morphism is of $\mathsf{finite\ type}$ if each of these $f^{-1}(\mathrm{Spec}\ B_i)$ are quasi-compact.

Definition. A morphism $f : X \to Y$ is said to be finite if Y can be covered by $\mathrm{Spec}\ B_i$ such that $f^{-1}(\mathrm{Spec}\ B_i) = \mathrm{Spec}\ A_i$ where A_i is a finitely generated B_i-module.

Proposition 2.19. *Let $f : X \to Y$ be a morphism of schemes.*

1. *X is locally noetherian if and only if for any affine open set $\mathrm{Spec}\ A$, A is noetherian.*

2. *f is locally finite type if and only if for any open affine subset $\mathrm{Spec}\ B$ of Y, $f^{-1}(\mathrm{Spec}\ B)$ can be covered by affine open subsets $\{\mathrm{Spec}\ A_i\}$ where each A_i is a finitely generated B-module.*

Proof. (1) [3, page 83, 84]
(2) One way is obvious. If for any open affine subset $\mathrm{Spec}\ B$ of Y, $f^{-1}(\mathrm{Spec}\ B)$ can be covered by affine open subsets $\{\mathrm{Spec}\ A_i\}$ where each A_i is a finitely generated B-module, then it is clear that it is of locally finite type. To prove the converse first note that since f is of locally finite type, there exists an open cover $\{\mathrm{Spec}\ B_i\}$ of Y which satisfy the neccessary condition. Now, the open subsets of the form $\mathrm{Spec}\ (B_i)_g$ will also satisfy the neccessary conditions for f to be locally finite type (Here $g \in B_i$). Since these form the basis for Y, we get that for every open affine set $U = \mathrm{Spec}\ B$, there exists an affine cover $\{\mathrm{Spec}\ A_i\}$ for $f^{-1}(\mathrm{Spec}\ B)$ such that each of these A_i is a finitely generated B-algebra. □

Proposition 2.20. *Let V be a variety over k. Then, $t(V)$ is integral, noetherian and finite type over k.*

Proof. [3, page 84] □

Definition. An $\mathtt{open\ subscheme}$ of a scheme X is scheme U such that $sp(U)$ is homeomorphic to an open subset of $sp(X)$ and $\mathcal{O}_U \cong \mathcal{O}_X|_U$. An $\mathtt{open\ immersion}$ is a morphism $f : Y \to X$ which induces an isomorphism with an open subscheme of X

Definition. A $\mathtt{closed\ immersion}$ is a morphism $f : Y \to X$ such that $sp(Y)$ is homeomorphic to its image, which is a closed subset of $sp(X)$ and the morphism $\mathcal{O}_X \to f_*\mathcal{O}_Y$ is surjetive. A $\mathtt{closed\ subscheme}$ of a scheme X is an equivalence class of closed immersions. Two closed immersions $f : Y \to X$ and $f' : Y' \to X$ are said to be equivalent if there exists a isomorphism $i : Y \to Y'$ such that $f = f' \circ i$.

Definition. Given a scheme X and a closed subset $Y \subseteq X$, we define the the $\mathtt{reduced\ induced\ subscheme}$ structure on Y. For any affine space $X = \mathrm{Spec}\ A$, where $Y = V(\mathfrak{a}) = V(\mathrm{rad}(\mathfrak{a}))$, give the reduced induced structure to Y defined by $\mathrm{rad}(\mathfrak{a})$. Now for any scheme X and closed subset Y, we can find a closed subscheme structure $\mathcal{O}_Y$ such that for each affine open subset $U = \mathrm{Spec}\ A$ of X, $Y \cap U$ with the structure sheaf $\mathcal{O}_Y|_{U\cap Y}$ is isomorphic to $\mathrm{Spec} A/\mathfrak{a}$ where $\mathfrak{a}$ is the radical ideal such that $V(\mathfrak{a}) = Y \cap U$ in U. This $\mathcal{O}_Y$ is called the reduced incduced subscheme structure on X.

Definition. Given morphisms $X \to S$ and $Y \to S$ of schemes, we define the fibred product of $X \times_S Y$ to be the fibred product of the diagram

$$\begin{tikzcd} & X \arrow[d] \\ Y \arrow[r] & S \end{tikzcd}$$

in the category of schemes.

Proposition 2.21. *Let $X \to S$ and $Y \to S$ be morphisms of scheme and $X \times_S Y$ the corresponding fibred product.*

1. *if $X = \mathrm{Spec}\ A$, $Y = \mathrm{Spec}\ B$ and $S = \mathrm{Spec}\ R$, then, $X \times_S Y \cong \mathrm{Spec}\ A \bigotimes_R B$;*
2. *for any open subscheme $U \subseteq X$, and $\pi_X^{-1}(U) = U \times_S Y$, where $\pi_X : X \times_S Y \to Y$.*

Proof. [3, page 87, 88] □

Definition. For any morphism $X \to S$ we define the $\mathtt{base\ extenison}$ of X with respect to a base morphism $S' \to S$ of schemes to be $X' = X \times_S S'$. For any scheme X, we define the n^{th} $\mathtt{projective\ space}$ over X, $\mathbb{P}^n_X$ to be the base extension of X with respect to $\mathbb{P}^n_{\mathbb{Z}} \to \mathrm{Spec}\ \mathbb{Z}$.

Proposition 2.22. *For any ring A, and $X = \mathrm{Spec}\ A$, $\mathbb{P}^n_X = \mathbb{P}^n_A$*

Proof. [3, page 120] □

Definition. A morphisms $X \to Y$ is said to be $\mathtt{projective}$ is there exists a closed immersion $X \to \mathbb{P}^n_Y$ such that the diagram commutes

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \\ & \mathbb{P}^n_Y \arrow[u] \end{tikzcd}$$

A morphisms $f : X \to Y$ is said to be $\mathtt{quasi\text{-}projective}$ if there exists an open immersion $g : X \to X'$ and a projective morphisms $h : X' \to Y$ such that $f = h \circ g$

Definition. Any morphism $f : X \to Y$ is said to be **separated** if the diagonal morphism $\Delta : X \to X \times_Y X$ is a closed embeding.

$$\begin{tikzcd}
X \arrow[dr, dotted, "\Delta"] \arrow[drr, bend left, "id_X"] \arrow[ddr, bend right, "id_X"'] & & \\
& X \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

We say a scheme is separated if it is separated over $\mathsf{Spec}\ \mathbb{Z}$

Proposition 2.23. *Any morphism $X \to Y$ is separated if and only if $\Delta(X)$ is a closed subset of $X \times_Y X$*

Proof. [3, page 96] □

Proposition 2.24. *Any morphism of affine schemes is separated.*

Proof. Let $X = \mathsf{Spec}\ B$ and $Y = \mathsf{Spec}\ A$ be affine schemes. And, $X \to Y$ be a morphism of schemes. Then , $X \times_Y X \cong \mathsf{Spec}\ (B \bigotimes_A B)$ and the morphism $\Delta : \mathsf{Spec}\ B \to \mathsf{Spec}\ (B \bigotimes_A B)$ is induced by the homomorphism of rings $B \bigotimes_A B \to B$ which takes $b \bigotimes b' \mapsto bb'$. Now, this homomorphism is clearly surjective. Thus, if $\mathfrak{a}$ is the kernel of the homomorphism, then $\Delta(X) = V(\mathfrak{a})$ which is closed by definiton and thus the morphism $X \to Y$ is separated. □

Proposition 2.25. *For any morphism $f : X \to Y$, $\Delta(X)$ is locally closed subscheme of $X \times_Y X$ (i.e. there exists an open subset $U \subseteq X \times_Y X$ such that $\Delta(X)$ is a closed subscheme of U). Moreover $\Delta(X) \cong X$.*

Proof. Let V_i be a open affine cover of Y and $U_{i,j}$ an open affine cover of $f^{-1}(V_i)$. Denote the natural restriction of f of $U_{i,j}$ by $f_{i,j} : U_{i,j} \to V_i$ Now, each of these $f_{i,j}$ are seperated as we saw above and the open subsets $U_{i,j} \times_{V_i} U_{i,j} \subseteq X \times_Y X$ covers $\Delta(X)$. Now, $\Delta(X) \cap (U_{i,j} \times_{V_i} U_{i,j}) = \Delta_{i,j}(U_{i,j})$ which is closed in $U_{i,j} \times_{V_i} U_{i,j}$. Thus, $\Delta(X)$ is closed in $U = \bigcup_{i,j}(U_{i,j} \times_{V_i} U_{i,j})$. □

Proposition 2.26. *Suppose all schemes considered bellow are noetherian*

1. *Open and closed imersions are separated*
2. *A composition of two separated morphisms is separated*
3. *if $f : X \to Y$ and $f' : X' \to Y'$ are separated morphisms over a scheme S, then $f \times f' : X \times_S X' \to Y \times_S Y'$ is also separated.*
4. *If $f : X \to Y$ and $g : Y \to Z$ are two morphisms and if $g \circ f$ is separated them f is separated.*

Proof. [3, page 99] □

Proposition 2.27. *For any algebraically closed field k, the image of the functor from the category of varieties over k to the category of schemes over k is exactly the quasi projective schemes over k. And the image of projective varieties over k is exactly the projective schemes over k. In particular, the image of any variety over k is an integral, separated scheme of finte-type over k.*

Proof. [3, page 104] □

Definition. An **abstract variety** is defined to be an integral separated scheme of finte-type over k.

From now on, the term variety will mean abstract variety unless otherwise specified.

2.3 Sheaves of Modules

Definition. Given a ringed space $(X, \mathcal{O}_X)$, we define a **sheaf of $\mathcal{O}_X$-modules** to be a sheaf $\mathcal{F}$ of abelian groups on X such that for each open set $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module and for any pair of open sets $U \subseteq V$, the maps $\mathcal{F}(V) \to \mathcal{F}(U)$ and $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ are compatible.

Definition. For any ringed space X and $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, a **morphisms of $\mathcal{O}_X$-modules** is a sheaf morphism $f : \mathcal{F} \to \mathcal{G}$ such that for each open set U, the map $f(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is an $\mathcal{O}_X(U)$-module homomorphism. The set of $\mathcal{O}_X$-module homorphisms is denoted by $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ or $\mathrm{Hom}_X(\mathcal{F}, \mathcal{G})$. The presheaf $U \mapsto \mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ is in fact a sheaf and denoted by $\mathscr{H}\!om_X(\mathcal{F}, \mathcal{G})$.

The kernel, cokernel and image of any morphism of $\mathcal{O}_X$-modules is an $\mathcal{O}_X$-module. For any pair of $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, we define the tensor product $\mathcal{F} \bigotimes_{\mathcal{O}_X} \mathcal{G}$ to be the sheaf associated to the presheaf $U \mapsto \mathcal{F}(U) \bigotimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$.

Proposition 2.28. *For any ringed space X and $\mathcal{O}_X$-module $\mathcal{F}$, $\mathscr{H}\!om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \cong \mathcal{F}$*

Proof. It will be adequate to show that $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$ since this would imply that $\mathrm{Hom}_{\mathcal{O}_X|_U}(\mathcal{O}_X|_U, \mathcal{F}|_U) \cong \Gamma(U, \mathcal{F})$ and thus we would be done. We have a natural homorphism $\Gamma(X, \mathcal{F}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F})$ where for each $s \in \mathcal{F}(X)$, we have $1 \in \mathcal{O}_X(X)$ going to s and the image of every other section will follow and therefore we have a morphism $\phi_s : \mathcal{O}_X \to \mathcal{F}$. This map by definition is clearly injective. Now, let $\phi : \mathcal{O}_X \to \mathcal{F}$ be a morphism. And let $t \in \mathcal{F}(X)$ such that $\phi(X)(1) = t$. Since the image of every section depends on the image of 1 in $\mathcal{O}_X(X)$, $\phi = \phi_t$. Thus, the homomorphism $\Gamma(X, \mathcal{F}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F})$ is in fact an isomorphism. □

Definition. We say that an $\mathcal{O}_X$-module $\mathcal{F}$ is **free** if $\mathcal{F} \cong \bigoplus_{j \in J} \mathcal{O}_X$ and is called **locally free** if X can be covered by open sets $\{U_i\}$ such that $\mathcal{F}|_{U_i} \cong \bigoplus_{j \in J_i} \mathcal{O}_X|_{U_i}$, where J and J_i are indexing sets. The **rank** of a localy free sheaf on each of these U_i is the cardinality of J_i and if X is connected then it is the same thoroughout and is called the rank of $\mathcal{F}$. A locally free sheaf of rank 1 is called an **invertible sheaf**.

For any morphism $f : X \to Y$ of ringed spaces and for any $\mathcal{O}_X$-module $\mathcal{F}$, we define the sheaf $f^*\mathcal{F} = f^{-1}\mathcal{F} \bigotimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$. Note that $\mathcal{O}_X$ has a natural $f^{-1}\mathcal{O}_Y$-module structure. Now there is a natural isomorphism of groups $\mathrm{Hom}_X(f^*\mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_Y(\mathcal{F}, f_*\mathcal{G})$ where $\mathcal{F}$ is an $\mathcal{O}_Y$-module and $\mathcal{G}$ is an $\mathcal{O}_X$-module.

Definition. Given any locally free $\mathcal{O}_X$-module $\mathcal{E}$ of finite rank, we define its dual $\check{\mathcal{E}}$ to be $\mathscr{H}\!om(\mathcal{E}, \mathcal{O}_X)$.

Proposition 2.29. *Let X be a ringed space and $\mathcal{E}$ locally free $\mathcal{O}_X$-module of finite rank. Then:*

1. $(\check{\mathcal{E}})\check{} \cong \mathcal{E}$

2. *For any* $\mathcal{O}_X$*-module* $\mathcal{F}$, $\mathscr{H}om(\mathcal{E},\mathcal{F}) \cong \check{\mathcal{E}} \bigotimes \mathcal{F}$;

3. *For any* $\mathcal{O}_X$*-modules* $\mathcal{F},\mathcal{G}$, $\mathsf{Hom}(\mathcal{E} \bigotimes \mathcal{F},\mathcal{G}) \cong \mathsf{Hom}(\mathcal{F},\mathscr{H}om(\mathcal{E},\mathcal{G}))$;

4. **Projective formula.** *If* $f : X \to Y$ *is a morphism of ringed spaces, if* $\mathcal{F}$ *is an* $\mathcal{O}_X$*-module, and if* $\mathcal{E}$ *is a locally free* $\mathcal{O}_Y$*-module of finite rank, then there is a natural isomorphism* $f_*(\mathcal{F} \bigotimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong (f_*\mathcal{F}) \bigotimes_{\mathcal{O}_Y} \mathcal{E}$

Proof. *(1)* Let us first define the natural morphism $\phi : \mathcal{E} \to (\check{\mathcal{E}})\check{} \cong \mathscr{H}om_{\mathcal{O}_X}(\mathscr{H}om_{\mathcal{O}_X}(\mathcal{E},\mathcal{O}_X),\mathcal{O}_X)$. Let $U \subseteq X$ be an open set. Then, $\phi_U : \mathcal{E}(U) \to \mathsf{Hom}(\mathscr{H}om(\mathcal{E},\mathcal{O}_x)|_U,\mathcal{O}_X|_U)$ such that for any section $s \in \mathcal{E}(U)$, $\phi_U(s)$ is the following morphism from $\phi_{U,s} : \mathscr{H}om(\mathcal{E},\mathcal{O}_x)|_U \to \mathcal{O}_X|_U$: For any open set $V \subseteq U$, $\phi_{U,s}(V) : \mathsf{Hom}(\mathcal{E}|_V,\mathcal{O}_X|_V) \to \mathcal{O}_X(V)$ which takes any morphism $\psi \in \mathsf{Hom}(\mathcal{E}|_V,\mathcal{O}_X|_V)$ to $\psi_V(s|_V)$. Since ismomorphism is a local condition, to prove that the above map is an isomorphism, we can in fact assume that $\mathcal{E}$ is free. Now, there is a natural isomorphism $\mathsf{Hom}_{\mathcal{O}_X}(\bigoplus \mathcal{O}_X,\mathcal{G}) \cong \mathsf{Hom}_{\mathcal{O}_X(X)}(\bigoplus \mathcal{O}_X(X),\mathcal{G}(X))$ for any $\mathcal{O}_X$-module $\mathcal{G}$. Thus, since $\mathcal{E}$ is free and since we know that for any free A-module M, $M^{**} \cong M$, the above map ϕ is in fact an isomorphism.
(2) Like above, let us define a morphism $\phi : \check{\mathcal{E}} \bigotimes \mathcal{F} \to \mathscr{H}om(\mathcal{E},\mathcal{F})$. Let U be an open set $\phi_U : \mathsf{Hom}_{\mathcal{O}_X|_U}(\mathcal{E}|_U,\mathcal{O}_X|_U) \bigotimes \mathcal{F}(U) \to \mathsf{Hom}_{\mathcal{O}_X|_U}(\mathcal{E}|_U,\mathcal{F}|_U)$. For any morphism $\psi \in \mathsf{Hom}_{\mathcal{O}_X|_U}(\mathcal{E}|_U,\mathcal{O}_X|_U)$ and $m \in \mathcal{F}(U)$, $\psi \bigotimes m$ goes to the morphism $\phi_{U,\psi \otimes m} : \mathcal{E}|_U \to \mathcal{F}|_U$ which for any open set $V \subseteq U$ takes any section $s \in \mathcal{E}(V)$ to $\psi_V(s) \cdot m|_V$. Now, this is a presheaf morphisms and we need to show that $\mathscr{H}om(\mathcal{E},\mathcal{F})$ is the sheaf associated to the presheaf $\check{\mathcal{E}} \bigotimes \mathcal{F}$. Let $\mathcal{G}$ be any sheaf on X and suppose $f : \check{\mathcal{E}} \bigotimes \mathcal{F} \to \mathcal{G}$ is a morphism. We define the corresponding unique morphism $\theta : \mathscr{H}om(\mathcal{E},\mathcal{F}) \to \mathcal{G}$. Once we define this morphism assuming $\mathcal{E}$ is free then we can define it for locally free $\mathcal{E}$ by defining it on the open sets on which it is free and then glueing these morphisms together. Therefore, let us assume $\mathcal{E} = \bigoplus_{i=1}^n \mathcal{O}_X$. Let $U \subseteq X$ be an open set. We first define morphims $\epsilon_i : \bigoplus_{i=1}^n \mathcal{O}_X|_U \to \mathcal{O}_X|_U$ which is defined by taking the element $e_i \in \bigoplus_{i=1}^n \mathcal{O}_X(U)$ to 1 and e_j to 0 whenever $i \neq j$ (Here $\{e_1,...,e_n\}$ is the standard generating set of $\bigoplus_{i=1}^n \mathcal{O}_X(U)$). Now, define the map $\theta_U : \mathsf{Hom}(\mathcal{E}|_U,\mathcal{F}|_U) \to \mathcal{G}(U)$. Consider any morphism $\alpha \in \mathsf{Hom}(\mathcal{E}|_U,\mathcal{F}|_U) \cong \mathsf{Hom}(\bigoplus_{i=1}^n \mathcal{O}_X|_U,\mathcal{F}|_U)$. Suppose $\alpha_U(e_i) = m_i$. Now let $\theta_U(\alpha) = f_U(\sum_{i=1}^n \epsilon_i \bigotimes m_i)$. This gives us the neccessary morphism and completes the proof.
(3) This is a consequnce of the commutative algebra result $\mathsf{Hom}(M \bigotimes N, P) \cong \mathsf{Hom}(M,\mathsf{Hom}(N,P))$ where M, N and P are A-modules.
(4) Again, we can assume that $\mathcal{E}$ if free since, since if $\mathcal{E}$ is locally free, we can prove the result on an open cover on which it is free and then paste these open subschemes. First, by the definition it is clear that $f^*\mathcal{E} \cong f^*(\bigoplus_{i=1}^n \mathcal{O}_Y) \cong \bigoplus_{i=1}^n \mathcal{O}_X$. Now, $f_*(\mathcal{F} \bigotimes_{\mathcal{O}_X} f^*\mathcal{E}) \cong f_*(\mathcal{F} \bigotimes_{\mathcal{O}_X}(\bigoplus_{i=1}^n \mathcal{O}_X)) \cong f_*(\bigoplus_{i=1}^n \mathcal{F}) \cong f_*\mathcal{F} \bigotimes_{\mathcal{O}_Y}(\bigoplus_{i=1}^n \mathcal{O}_Y) \cong f_*\mathcal{F} \bigotimes_{\mathcal{O}_Y} \mathcal{E}$. □

For any A-module M, we define a $\mathcal{O}_X$-module $\widetilde{M}$ where $X = \mathsf{Spec}\ A$ as follows. For any open set $U \subseteq X$, we define $\widetilde{M}(U)$ as the set of all function $s : U \to \coprod_{p \in U} M_p$ such that for all $p \in U$, $s(p) \in M_p$ and there exists an open set $V \subseteq U$ containing p and $m \in M$ and $f \in A$ such that for all $q \in V$, $f \notin q$ and $s(q) = {}^m/_f$

Proposition 2.30. *For any affine scheme* $X = \mathsf{Spec}\ A$ *and* A*-module* M, *let* $\widetilde{M}$ *be the* $\mathcal{O}_X$*-module associated to* M. *Then,*

1. *for each* $p \in X$, *the stalk* $(\widetilde{M})_p \cong M_p$;

2. *for each* $f \in A$, $\widetilde{M}(D(f)) \cong M_f$;

3. *As a special case of the above result,* $\Gamma(X, \widetilde{M}) = M$

Proof. [3, page 110] □

Proposition 2.31. *Let* A, B *be rings,* $A \to B$ *a ring homomorphism and* $f : \mathrm{Spec}\ B \to \mathrm{Spec}\ A$ *the corresponding morphism of affine shcemes. Then,*

1. *The functor* $M \to \widetilde{M}$ *gives an exact, fully faithful functor from the category of* A*-modules to the category of* $\mathcal{O}_X$*-modules, where* $X = \mathrm{Spec}\ A$.

2. *if* M *and* N *are* A*-modules, then* $\widetilde{M \bigotimes_A N} \cong \widetilde{M} \bigotimes_{\mathcal{O}_X} \widetilde{N}$;

3. *for any family* $\{M_i\}$ *of* A*-modules,* $\widetilde{\bigoplus M_i} \cong \bigoplus \widetilde{M_i}$;

4. *for any* B*-module* N, $f_*(\widetilde{N}) \cong \widetilde{N_A}$, *which is* N *considered as an* A*-module using the homomorphism* $A \to B$;

5. *for any* A*-module* M, $f^*(\widetilde{M}) = \widetilde{M \bigotimes_A B}$, *where* $M \bigotimes_A B$ *is considered as a* B*-module.*

Proof. [3, page 110, 111] □

Definition. For any scheme X, we call an $\mathcal{O}_X$-module $\mathcal{F}$ $\mathsf{quasi\text{-}coherent}$ if there exists an affine open cover $\{U_i = \mathrm{Spec}\ A_i\}$ of X and A_i-modules M_i such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$. We call it $\mathsf{coherent}$ each of these M_i are finitely generated.

Proposition 2.32. *Given a scheme* X *and a quasi-coherent sheaf* $\mathcal{F}$ *and a affine open subset* $\mathrm{Spec}\ A = U \subseteq X$, $\mathcal{F}|_U \cong \widetilde{M}$ *for some* A*-module* M. *If* X *is noetherian and* $\mathcal{F}$ *is coherent, then the above mentioned* M *is a finitely generated* A*-module.*

Proof. [3, page 113] □

Proposition 2.33. *Let* A *be a ring and let* $X = \mathrm{Spec}\ A$. *Then, the functor* $M \mapsto \widetilde{M}$ *gives an equivalence of categories betweent the category of* A*-modules and category of quasi-coherent sheaves over* X. *If* A *is noetherian, then the same functor gives an equivalence of categories between the category of finitely generated* A*-modules and coherent sheaves over* X.

Proof. [3, page 113] □

For reason as dipicted in the above proposition, we generally talk of coherent sheaves only when the the considered scheme is noetherian.

Proposition 2.34. *Let* X *be a scheme. The kernel, cokernel and image of a morphism of quasi-coherent sheaves is quasi-coherent. If* X *is noetherian, the same is true for coherent.*

Proof. [3, page 114] □

Proposition 2.35. *Let* X *be a noetherian scheme and* $\mathcal{F}$ *be a coherent sheaf over* X. *Then,* $\mathcal{F}$ *is locally free if and only if for each* $x \in X$, $\mathcal{F}_x$ *is a free* $\mathcal{O}_{X,x}$*-module.*

Proof. If $\mathcal{F}$ is locally free, then for any $x \in X$ we will have a neighbourhood $U \subseteq X$ containing x such that $(U, \mathcal{F}|_U)$ is free, This will immediately give us the result that $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}-\mathtt{module}$. Now, to prove the converse, we will show that if $\mathcal{F}_p$ is a free $\mathcal{O}_{X,p}$-module, then there is a neighbourhood U of p such that $\mathcal{F}|_U$ is free. Since the question is localy, we can assume that $X = \mathsf{Spec}\ A$ and $\mathcal{F} = \widetilde{M}$ where M is finitely generated. Now, $\mathcal{F}_p = M_p \cong \bigoplus_{i=1}^n A_p$. We can have a surjective morphism $\bigoplus_{i=1}^n A \to M \to 0$ such that its localisation gives the isomorphism $\bigoplus_{i=1}^n A_p \xrightarrow{\sim} M_p$. If K is the kernel of that surjective morhpism, we know that K is also finitely generated since A is noetherian. Now, $K_p = 0$. Thus, there exists an f such that $K_f = 0$ (since it is finitely genrated). Therefore, we have an isomorphims $M_f \xrightarrow{\sim} \bigoplus_{i=1}^n A_f$. And Thus, $\mathcal{F}|_{D(f)}$ is a free $\mathcal{O}_X|_{D(f)}$-module. This completes the proof. □

Proposition 2.36. *Let X be a noetherian scheme and $\mathcal{F}$ be a coherent sheaf over X. Then, $\mathcal{F}$ is invertible if and only if there exists a coherent sheaf $\mathcal{G}$ such that $\mathcal{F} \bigotimes \mathcal{G} \cong \mathcal{O}_X$ (this explain the useage of the term invertible). Infact, $\mathcal{G} = \check{\mathcal{F}}$*

Proof. If $\mathcal{F}$ is invertible, then we have a natural isomorphims $\mathcal{F} \bigotimes \check{\mathcal{F}} \xrightarrow{\sim} \mathcal{O}_X$ using the fact that $\mathcal{F} \bigotimes \check{\mathcal{F}} \cong \mathscr{H}om(\mathcal{F}, \mathcal{F})$. For the converse, it is adequate to show that the result is true locally (because we only need to show that the stalk is free of rank 1). Let $X = \mathsf{Spec}\ A$ and suppose for some coherent sheaf $\mathcal{F} = \widetilde{M}$, there exists a coherent sheaf $\mathcal{G} = \widetilde{N}$ such that $\mathcal{F} \bigotimes \mathcal{G} \cong \mathcal{O}_X$. Thus, for each $p \in X$, $M_p \bigotimes N_p \cong A_p$. Therefore, using Nakayama lemma we get that $M_p \cong N_p \cong A_p$. □

Proposition 2.37. *Let $f : X \to Y$ be a morphisms of schemes. Then,*

1. *If $\mathcal{G}$ is a quasi-coherent sheaf of $\mathcal{O}_Y$-modules, then $f^*\mathcal{G}$ is a quasi-coherent $\mathcal{O}_X$-module.*
2. *If X and Y are noetherian and if $\mathcal{G}$ is a coherent $\mathcal{O}_Y$-module, then $f^*\mathcal{G}$ is coherent $\mathcal{O}_X$-module.*
3. *If X is noetherian and if $\mathcal{F}$ is quasi-coherent sheaf of $\mathcal{O}_X$-modules, $f_*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-module.*

Proof. [3, page 115] □

Definition. Given a scheme X, we define an $\mathtt{ideal\ sheaf}$ as an $\mathcal{O}_X$-module $\mathscr{I}$ such that for each open set U, $\mathscr{I}(U)$ is an ideal of $\mathcal{O}_X(U)$. For any closed immersion $Y \to X$, we define the ideal sheaf on X with respect to Y (denoted as $\mathscr{I}_Y$) as the kernel of the morphism $\mathcal{O}_X \to f_*\mathcal{O}_Y$.

Every ideal sheaf is in fact an ideal sheaf with respect to a unique closed subscheme. Therefore for any affine scheme $\mathsf{Spec}\ A$ there is a one to one correspondence between its ideal sheaves of $\mathsf{Spec}\ A$ and ideals of A.
For any graded ring S and graded module M we define $\widetilde{M}$ over $\mathsf{Proj}\ S$ as follows. For any open set $U \subseteq \mathsf{Proj}\ S$, we define $\widetilde{M}(U)$ to be the set of all functions $s : U \to \coprod_{p \in U} M_{(p)}$ such that for each $p \in U$, $s(p) \in M_{(p)}$ and there exists an open set $V \subseteq U$ containing p and $m \in M$ and $f \in S$ homogeneous elements of the same degree such that for all $q \in V$, $f \notin q$ $s(q) = {}^m/_f$.

Proposition 2.38. *For a graded ring S and graded module M, let $\widetilde{M}$ be as defined above. Then,*

1. *For any $p \in \mathsf{Proj}\ S$, $(\widetilde{M})_P \cong M_{(p)}$;*

2. *For any homogeneous* $f \in S_+$, $\widetilde{M}|_{D(f)} \cong \widetilde{M_{(f)}}$.

3. $\widetilde{M}$ *is a quasi coherent* $\text{Proj } S$*-module and if* S *is noetherian and* M *is finitely generated, then* $\widetilde{M}$ *is coherent.*

Proof. [3, page 116, 117] □

Definition. For a graded ring S let $X = \text{Proj } S$ and $n \in \mathbb{Z}$. We define the $\mathcal{O}_X$-module $\mathcal{O}_X(n) = \widetilde{S(n)}$. For any $\mathcal{O}_X$-module $\mathcal{F}$, we define the n^{th} **twisted sheaf** of $\mathcal{F}$, $\mathcal{F}(n) = \mathcal{F} \bigotimes_{\mathcal{O}_X} \mathcal{O}_X(n)$

Proposition 2.39. *Let* S *be a graded ring and* $X = \text{Proj } S$*. Assume that* S *is generated by* S_1 *as an* S_0*-algebra. Then,*

1. *For any* $n \in \mathbb{Z}$, $\mathcal{O}_X(n)$ *is an invertible sheaf;*

2. *For any graded* S*-module* M, $\widetilde{M}(n) \cong \widetilde{M(n)}$*. Thus,* $\mathcal{O}_X(n) \bigotimes \mathcal{O}_X(m) \cong \mathcal{O}_X(n+m)$*;*

3. $\check{\mathcal{O}_X}(\mathrm{n}) \cong \mathcal{O}_X(-n)$

Proof. Part *(1)* and *(2)* are proved in [3, page 117]]
(3). We will be done if we show that $\check{\mathcal{O}_X}(1) \cong \mathcal{O}_X(-1)$. Because, for any positive n, we will have $\check{\mathcal{O}_X}(\mathrm{n}) \cong \mathscr{H}om(\mathcal{O}_X(n), \mathcal{O}_X) \cong \mathscr{H}om(\mathcal{O}_X(n-1), \check{\mathcal{O}_X}(1)) \cong \mathscr{H}om(\mathcal{O}_X(n-1), \mathcal{O}_X(-1))$ and recusively following this process, we will get $\check{\mathcal{O}_X}(\mathrm{n}) \cong \mathscr{H}om(\mathcal{O}_X(n), \mathcal{O}_X) \cong \mathscr{H}om(\mathcal{O}_X, \mathcal{O}_X(-n)) \cong \mathcal{O}_X(-n)$. And for any negative n, we have $\check{\mathcal{O}_X}(\mathrm{n}) \cong (\mathcal{O}_X(-n))\check{} \cong \mathcal{O}_X(-n)$ and we will be done. Now we define the following homomorphism $\phi : \mathcal{O}_X(-1) \to \mathscr{H}om(\mathcal{O}_X(1), \mathcal{O}_X)$. Let U be an open set. Now define the map $\phi(U) : \mathcal{O}_X(-1)(U) \to \text{Hom}_{\mathcal{O}_X|_U}(\mathcal{O}_X(1)|_U, \mathcal{O}_X|_U)$ Let $s \in \mathcal{O}_X(-1)(U)$. Now, $s : U \to \coprod_{p \in U} S(-1)_{(p)}$. We define the morphism from $\mathcal{O}_X(1)|_U$ to $\mathcal{O}_X|_U$ by multiplying each section by s restricted to that open set (since at the level of stalks each section is of degree 1, we we multiply by s-which at the level of stalks have degree -1, and thus we get a section of $\mathcal{O}_X$). By checking at the level of stalk, we get that this morphism is in fact an isomorphism. Thus, we get that $\check{\mathcal{O}_X}(1) = \mathscr{H}om(\mathcal{O}_X(1), \mathcal{O}_X) \cong \mathcal{O}_X(-1)$. □

Definition. Let S be a graded ring and $X = \text{Proj } S$ and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. We define a graded module with respect to $\mathcal{F}$, $\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n))$. For any $s \in S_d$ and $m \in \Gamma(X, \mathcal{F}(n))$, then we define $s \cdot m \in \Gamma(X, \mathcal{F}(n+d)) = \Gamma(X, \mathcal{O}_X(n+d) \bigotimes \mathcal{F}) \cong \Gamma(X, \mathcal{O}_X(d) \bigotimes \mathcal{O}_X(n) \bigotimes \mathcal{F}) \cong \Gamma(X, \mathcal{O}_X(d) \bigotimes \mathcal{F}(n))$ by taking the tensor product $s \bigotimes m \in \Gamma(X, \mathcal{O}_X(d) \bigotimes \mathcal{F}(n))$

Proposition 2.40. *Let* S *be a graded ring which is finitely generated by* S_1 *as an* S_0*-algebra and* $X = \text{Proj } S$*. Let* $\mathcal{F}$ *be a quasi coherent* $\mathcal{O}_X$*-module, then* $\mathcal{F} \cong \widetilde{\Gamma_*(\mathcal{F})}$

Proof. [3, page 119] □

Proposition 2.41. *A scheme* Y *over* $\text{Spec } A$ *is projective if and only if it is isomorphic to* $\text{Proj } S$ *for some graded ring* S *where* $S_0 = A$*, and* S *is finitely generated by* S_1 *as an* S_0*-algebra.*

Proof. [3, page 119, 120] □

Definition. For any scheme Y, we define a the $\mathtt{twisting\ sheaf}$ $\mathcal{O}(1)$ on $\mathbb{P}^r_Y$ to be $g^*\mathcal{O}(1)$, where $g : \mathbb{P}^r_Y \to \mathbb{P}^r_{\mathbb{Z}}$ is the natural map. Let X be an scheme over Y. An invertible sheaf $\mathcal{I}$ on X is $\mathtt{very\ ample}$ realative to Y, if there is an immersion $i : X \to \mathbb{P}^r_Y$ such that $\mathcal{I} \cong i^*\mathcal{O}(1)$. Here, we define an $\mathtt{immersion}$ as a morphism $X \to Z$ which gives an isomorphism to an open subscheme of a closed subsheme of Z.

Definition. We say that a sheaf of modules $\mathcal{F}$ on some scheme X is generated by global section if there exists a set $G \subseteq \Gamma(X, \mathcal{F})$ such that for each $x \in X$, $\mathcal{F}_x$ is generated by the image of elements in G. We say that it can be generated by a $\mathtt{finite\ number}$ of global section if we find a finite set G satifying the above property. Moreover, $\mathcal{F}$ is a sheaf of modules generated by global section if and only if it can be written as a quotient of a free sheaf.

Proposition 2.42. *Let X be a projective scheme over a noetherian ring A, let $\mathcal{O}(1)$ be a very ample invertible sheaf on X and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module.Then there is an integer n_0 such that for all $n \geq n_0$, the sheaf $\mathcal{F}(n)$ can be generated by a finite number of global sections.*

Proof. [3, page 121] □

Definition. An invertible sheaf $\mathcal{I}$ on a noetherian scheme X is said to be $\mathtt{ample}$ if for every coherent sheaf $\mathcal{F}$ on X, there exists an integer $n_0 > 0$ such that for every $n \geq n_0$, $\mathcal{F} \bigotimes \mathcal{I}^n$ is generated by its global sections. Here, $\mathcal{I}^n$ denoted tensor product n times with itself.

Proposition 2.43. *Let X be projective over a noetherian ring A. Then any coherent sheaf $\mathcal{F}$ on X can be written as a quotient of a sheaf $\mathcal{E}$, where $\mathcal{E}$ is a finite direct sum of twisted structure sheaves $\mathcal{O}(n_i)$ for various integers n_i.*

Proof. [3, page 121, 122] □

Definition. Let $X \to Y$ be a morphism. We define the $\mathtt{sheaf\ of\ relative\ differentials}$ of X over Y, $\Omega_{X/Y}$. Let $\Delta : X \to X \times_Y X$ be the diagonal morphism. Now, there exists an open subscheme $W \subseteq X \times_Y X$ such that $\Delta(X) \subseteq W$ is a closed subscheme. Moreover, $\Delta(X) \cong X$. Let $\mathscr{I}$ be the ideal sheaf of $\Delta(X)$ in W. We define $\Omega_{X/Y} = \Delta^*({}^{\mathscr{I}}/_{\mathscr{I}^2})$.

For any open affine set $\mathrm{Spec}\ A = V \subseteq Y$ and $\mathrm{Spec}\ B = U \subseteq X$ such that in the morphism $f : X \to Y$, $U \subseteq f^{-1}(V)$. Then, $\Omega_{U/V} \cong \Omega_{B/A}$. Thus, the sheaf of relative differentials can also be created by taking such an affine covers. The natural map $d : \mathcal{O}_X \to \Omega_{X/Y}$ is the differential map such that for each $x \in X$, $(\Omega_{X/Y})_x \cong \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{Y,f(x)}}$ and the map $d_x : \mathcal{O}_{X,x} \to (\Omega_{X/Y})_x$ is the corresponding $\mathcal{O}_{Y,f(x)}$-derivation.

Proposition 2.44. *Let $X \to Y \to Y'$ be a sequence of morphisms. Let $X' = X \times_Y Y'$ and let $g : X' \to X$ be the natural morphisms. Then, $\Omega_{X'/Y'} \cong g^*(\Omega_{X/Y})$*

Proof. [3, page 175] □

Proposition 2.45. *Let $X \xrightarrow{f} Y \to Z$ be a sequence of exact sequence. Then, there is an exact sequence of $\mathcal{O}_X$-modules: $f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0$*

Proof. [3, page 176] □

Proposition 2.46. *Let A be a ring, let $Y = \mathrm{Spec}\ A$ and let $X = \mathbb{P}^n_A$. Then, there is an exact sequence of $\mathcal{O}_X$-modules:* $0 \to \Omega_{X/Y} \to \bigoplus_{j=1}^{n+1} \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0$

Proof. [3, page 176, 177] □

Definition. In a similar manner as with a module over a ring we define the the $\mathit{exterior\ algebra}$ of an $\mathcal{O}_X$-module $\mathcal{F}$, $\bigwedge(\mathcal{F}) = \bigoplus_{n\geq 0} \bigwedge^n(\mathcal{F})$, where $\bigwedge^n(\mathcal{F})$ is defined by taking the sheaf associated to the presheaf $U \mapsto \bigwedge^n(\mathcal{F}(U))$, where each of there exterior algebras are taken over the corresponding ring $\mathcal{O}_X(U)$.

Proposition 2.47. *Let X be a scheme and $\mathcal{F}$ and be a locally free $\mathcal{O}_X$-module of rank n. Then $\bigwedge^r \mathcal{F}$ is locally free of rank $\binom{n}{r}$.*

Proof. If we show that for some open set U on which $\mathcal{F}$ is free $(\bigwedge^r \mathcal{F})|_U$ is free of rank $\binom{n}{r}$ we are done. Therefore, it is enough to show that if $\mathcal{F}$ is free of rank n, then $\bigwedge^r \mathcal{F}$ is free of rank $\binom{n}{r}$. Let $M = \bigoplus_{i=1}^n A$ be a free A-module. Let $e_1, ..., e_n \in M$ be the standard generators of M. Then, $\bigwedge^r M$ is the free A-module generated by the elements $e_{\alpha_1} \bigotimes \cdots \bigotimes e_{\alpha_r}$ where $\alpha_i = \alpha_j$ if and only if $i = j$. Thefore $\bigwedge^r M$ is the free A-module of rank $\binom{n}{r}$. Therefore for any free sheaf $\mathcal{F}$ of rank n, $(\bigwedge^r \mathcal{F})(U)$ is a free $\mathcal{O}_X(U)$-module of rank $\binom{n}{r}$. This completes the proof. □

Definition. Let X be a variety over an algebraically closed field k. We say that X is $\mathit{non\text{-}singular}$ if for all $x \in X$, $\mathcal{O}_{X,x}$ is a regular local ring.

Definition. Let X be a non-singular variety over an algebraically closed field k of dimension n. Then, we define the $\mathit{canonical\ sheaf}$ over X to be $\omega_X = \bigwedge^n \Omega_{X/k}$.

From the above result we know that ω_X is locally free of rank $\binom{n}{n} = 1$.

Proposition 2.48. *Let k be a field and $X = \mathbb{P}^n_k$. Then, $\omega_X \cong \mathcal{O}(-n-1)$*

Proof. Recall the eact sequence $0 \to \Omega_{X/k} \to \bigoplus_{j=1}^{n+1} \mathcal{O}_X(-1) \to \mathcal{O}_X \to 0$. Now, $\bigwedge^n \Omega_{X/k} = \omega_X$, $\bigwedge^n \mathcal{O}_X = 0$ and finally, $\bigwedge^n(\bigoplus_{j=1}^{n+1} \mathcal{O}_X(-1)) \cong \mathcal{O}_X(-1) \bigotimes \cdots \bigotimes \mathcal{O}_X(-1) = \mathcal{O}_X(-n-1)$. Thus, if we take the wedge product in the above exact sequence, we get a new exact sequence $0 \to \omega_X \to \mathcal{O}_X(-n-1) \to 0$. Therefore, we have an isomorphims $\omega_X \cong \mathcal{O}_X(-n-1)$. □

Proposition 2.49. *Any nonsingular variety is Cohen-Macaulay.*

Proof. Any local ring of a nonsingular variety is noetherian, local and regular. Therefore, every local ring is also Cohen-Macaulay. Thus, the variety is Cohen-Macaulay. □

Chapter 3

Cohomology

Like the previous chapter, the proofs that are well written in Hartshorne are cited here and incase these proofs are possibly not descriptive enough, then they shall be rewritten here with more details.

3.1 Sheaf Cohomology

Proposition 3.1. *For any ringed space $(X, \mathcal{O}_X)$, the category $\mathscr{Mod}(X)$ of $\mathcal{O}_X$-modules has enough injectives.*

Proof. [3, page 207] □

Proposition 3.2. *If X is a topological space, then the category $\boldsymbol{Ab}(X)$ of sheaves of abelian groups on X has enough injectives.*

Proof. Let $\mathcal{O}_X$ be the constant sheaf of rings $\mathbb{Z}$. Since every abelian group is a $\mathbb{Z}$-module, $\mathsf{Ab}(X)$ is the same as the category $\mathscr{Mod}(X, \mathcal{O}_X)$ which, from the above proposition we already know has enough injectives □

Definition. For any topological space X, Let $\Gamma(X, \cdot) : \mathbf{Ab}(X) \to \mathbf{Ab}$ be the global sections functor. Then, we define the cohomology functor $H^i(X, \cdot)$ to be the right derived functor of $\Gamma(X, \cdot)$. For any sheaf $\mathcal{F}$ on X, we define $H^i(X, \mathcal{F})$ to be the i^{th} cohomology group of $\mathcal{F}$.

Proposition 3.3. *Let $\mathcal{F}$ be a sheaf on a topological space X. Then, $H^0(X, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$.*

Proof. We know that Given a left exact contavariant functor F, $R^0F \cong F$. Therefore, $H^0(X, \cdot) \cong \Gamma(X, \cdot)$ □

Proposition 3.4. *For any ringed space X, an injective object in $\mathscr{Mod}(X)$ is flasque.*

Proof. [3, page 207] □

Proposition 3.5. *Every flasque sheaf is acyclic.*

Proof. [3, page 208] □

Thus, for a ringed space X, one can also find the cohomology groups by considering the global sections functor $\Gamma(X, \cdot) : \mathscr{Mod}(X) \to \mathbf{Ab}$.

Proposition 3.6. *Let X be a noetherian topological space, and let $(\mathcal{F}_\alpha)$ be a directed system of abelian sheaves. Then, there are natural isomorphisms for each $i \geq 0$ $\varinjlim H^i(X, \mathcal{F}_\alpha) \cong H^i(X, \varinjlim \mathcal{F}_\alpha)$.*

Proof. [3, page 209] □

As a special case of the above result, we know that direct sum and cohomology group commute.

Proposition 3.7. *Let Y be a closed subset of a toplogical space X and let $\mathcal{F}$ be a sheaf of abelian groups on Y. Suppose $j : Y \to X$ is the inclusion map. Then $H^i(Y, \mathcal{F}) = H^i(X, j_*\mathcal{F})$ for all $i \geq 0$*

Proof. Suppose I be an flasque sheaf on Y. Let $U \subseteq X$ be an open set. Now, $j^{-1}(U) \subseteq Y = j^{-1}(X)$. Thus, $(j_*I)(X) \to (j_*I)(U)$ is surjective and thus j_*I is flasque. Next, suppose $\mathcal{G}' \to \mathcal{G} \to \mathcal{G}"$ is exact sequence of sheaves on Y. Since Y is closed, we know that $(j_*\mathcal{G}')_x, (j_*\mathcal{G})_x$ and $(j_*\mathcal{G}")_x$ are $\mathcal{G}'_x$, $\mathcal{G}_x$ and $\mathcal{G}"_x$ respectively whenever $x \in Y$ and is zero otherwise. Thus, if $0 \to \mathcal{F} \to I^0 \to I^1 \to \cdots$ is a flasque resolution of $\mathcal{F}$, then $0 \to j_*\mathcal{F} \to j_*I^0 \to j_*I^1 \to \cdots$ is a flasque resolution of $j_*\mathcal{F}$. Now, since $\Gamma(Y, \mathcal{G}) = \Gamma(X, j_*\mathcal{G})$, we can conclude that $H^i(Y, \mathcal{F}) = H^i(X, j_*\mathcal{F})$ for all $i \geq 0$ □

Proposition 3.8. *Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}" \to 0$ be an exact sequence of sheaves on a topological space X. Then, there is a long exact sequence $0 \to H^0(X, \mathcal{F}') \to H^0(X, \mathcal{F}) \to H^0(X, \mathcal{F}") \xrightarrow{\delta^0} H^1(X, \mathcal{F}') \to \cdots \to H^i(X, \mathcal{F}') \to H^i(X, \mathcal{F}) \to H^i(X, \mathcal{F}") \xrightarrow{\delta^i} H^{i+1}(X, \mathcal{F}') \to \cdots$*

Proof. This follows immediately from the fact that the cohomology groups are a right derived functor of a left exact functor. □

Proposition 3.9. Grothendieck Vanishing Theorem. *Let X be a noetherian topological space of dimension n. Then for all $i > n$ and all sheaves of abelian groups $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F}) = 0$.*

Proof. [3, page 210, 211] □

3.2 Čech Cohomology

Let X be a topological space and $\mathcal{F}$ a sheaf on X. Let $\mathcal{U} = \{U_i\}_{i \in I}$ be a fixed cover of X. We also fix a total ordering for I. Furthermore for any $i_0, \ldots i_n \in I$, we denote by $U_{i_0,\ldots,i_n} = U_{i_0} \cap \cdots \cap U_{i_n}$. Now we define a chain complex for $p \geq 0$ as follows:

$$C^p(X, \mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0,\ldots,i_p})$$

Since it is clear what X would be if $\mathcal{U}$ is given, we often denote $C^p(X, \mathcal{U}, \mathcal{F})$ by simply writing $C^p(\mathcal{U}, \mathcal{F})$. Now, for any $\alpha \in C^p(\mathcal{U}, \mathcal{F})$, we denote it's corresponding elements in $\mathcal{F}(U_{i_0,\ldots i_p})$ by $\alpha_{i_0,\ldots,i_p}$. Now, we define the map $d^p : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ as

$$(d^p\alpha)_{i_0,\ldots,i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0,\ldots,\widehat{i_k},\ldots,i_{p+1}}|_{U_{i_0,\ldots,i_{p+1}}}$$

Here, the notation $\widehat{i_k}$ means omit i_k.

Now we have the Čech Complex $0 \to C^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d^0} C^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d^1} \cdots \xrightarrow{d^{p-1}} C^p(\mathcal{U}, \mathcal{F}) \xrightarrow{d^p} \cdots$ which we denote by $C^\bullet(\mathcal{U}, \mathcal{F})$. Now, for each $i \geq 0$, we define the i^{th} Čech cohomology group as $\check{H}^i(\mathcal{U}, \mathcal{F}) = h^i(C^\bullet(\mathcal{U}, \mathcal{F}))$

Proposition 3.10. *Let X be a topological space and $\mathcal{U}$ be an open covering. Then, for all $p \geq 0$ there is a natural map, functorial in $\mathcal{F}$, $\check{H}^i(\mathcal{U}, \mathcal{F}) \to H^i(X, \mathcal{F})$*

Proof. [3, page 221] □

Proposition 3.11. *Let X be a noetherian separated scheme and let $\mathcal{U}$ be an open affine cover of X and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then, for all $p \geq 0$, $\check{H}^i(\mathcal{U}, \mathcal{F}) \cong H^i(X, \mathcal{F})$*

Proof. [3, page 222] □

3.3 Cohomology of Projective schemes

Proposition 3.12. *Let A be a noetherian ring and let $X = \mathbb{P}^r_A$, with $r \geq 1$. Let $S = A[x_0, ..., x_r]$. Then:*

1. *The natural homomorphism $S \to \Gamma_*(\mathcal{O}_X) = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{O}_X(n))$ is an isomorphism of graded S-modules;*
2. *$H^i(X, \mathcal{O}_X(n)) = 0$ for $0 < i < r$ and all $n \in \mathbb{Z}$*
3. *$H^r(X, \mathcal{O}_X(-r-1)) \cong A$*
4. *$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \to H^r(X, \mathcal{O}_X(-r-1)) \cong A$ is a perfect paring of finitely generated free A-modules, for each $n \in \mathbb{Z}$*

Proof. [3, page 226, 227, 228] □

Proposition 3.13. *Let X be a projective scheme over a noetherian ring A and let $\mathcal{O}_X(1)$ be a very ample invertible sheaf on X over $\operatorname{Spec} A$. Let $\mathcal{F}$ be a coherent sheaf on X. Then,*

1. *For each $i \geq 0$, $H^i(X, \mathcal{F})$ is finitely generated free module;*
2. *There is an integer n_0 (depending on $\mathcal{F}$), such that for all $i > 0$ and $n \geq n_0$, $H^i(X, \mathcal{F}(n)) = 0$*

Proof. (1) Firstly, $\mathcal{O}_X(1) = j^*\mathcal{O}_P(1)$, where $P = \mathbb{P}^r_k$ for some r, where j is a closed immersion. Now, for any coherent sheaf $\mathcal{F}$ on X, $j_*\mathcal{F}$ is coherent on P and $H^i(X, \mathcal{F}) = H^i(P, j_*\mathcal{F})$. Thus, it will enough to show that the result is true for $X = \mathbb{P}^r_k$. From the cited proof of the above result, it is clear that $H^i(P, \mathcal{O}_P(q))$ is the module of all homogeneous polynomials of degree q (which is the zero module when $q < 0$) when $i = 0$; it is zero for all q when $0 < i < r$; it is the module of all polynomials with only negative powers adding up to degree q (which is the zero module when $q > -r-1$) when $i = r$; and lastly when $i > r$, it is zero by Grothendieck Vanishing Theorem. Thus $H^i(P, \mathcal{O}_P(q))$ is always a finitely generated A-module. Thus, if $\mathcal{E} = \bigoplus_{i=1}^N \mathcal{O}_P(q_i)$, then $H^i(X, \mathcal{E})$ is also a finitely generated A-module. We prove the result by induction. We know that $H^k(P, \mathcal{G})$ is a finitely generated A-module for all $k > r$ and $\mathcal{G}$ coherent. Let $i \in \mathbb{N} \cup \{0\}$ such that $H^k(P, \mathcal{G})$ is a finitely generated A-module of all $k > i$

and $\mathcal{G}$ coherent. Now, let $\mathcal{F}$ be a coherent sheaf on P, then there exists a sheaf of the form $\mathcal{E}$ as described above such that $0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0$ is an exact sequence of sheaves. Here, $\mathcal{R}$ is the kernel of the morphism and hence also coherent. Now, we have an exact sequence $\cdots \to H^i(P, \mathcal{E}) \to H^i(P, \mathcal{F}) \to H^{i+1}(P, \mathcal{R}) \to \cdots$. By, the hypothesis, we know that $H^{i+1}(P, \mathcal{R})$ is a finitely generated A-module and we have already shown that $H^i(P, \mathcal{E})$ is a finitely generated A-module. Thus, since A is noetherian, we have that $H^i(P, \mathcal{F})$ is a finitely generated A-module. Thus for any coherent sheaf $\mathcal{G}$, $H^i(P, \mathcal{G})$ is finitely generated A-module. Thus, by induction $H^k(P, \mathcal{G})$ is a finitely generated A-module for any coherent sheaf $\mathcal{G}$ and $k \geq 0$. This completes the proof.
(2) For any coherent sheaf $\mathcal{F}$ on X, we have $H^i(X, \mathcal{F}(n)) = H^i(P, j_*(\mathcal{F}(n))) =$
$H^i(P, j_*(\mathcal{F} \bigotimes j_*\mathcal{O}_P(n))) = H^i(P, j_*\mathcal{F} \bigotimes \mathcal{O}_P(n)) = H^i(X, (j_*\mathcal{F})(n))$ using the projective formula. Therefore, again it is adequate to prove the result for when $X = P$. Like before, we prove using induction. We know that $H^k(P, \mathcal{G}) = 0$ for all $k > r$. Let $i \in \mathbb{N}$ such that $H^{i+1}(X, \mathcal{G}) = 0$ for all coherent sheaves $\mathcal{G}$. Following from the same exact sequence as above $0 \to \mathcal{R} \to \mathcal{E} \to \mathcal{F} \to 0$, we first twist by some $n \in \mathbb{N}$ and then consider the long exact sequence.

$$\cdots \to H^i(P, \mathcal{E}(n)) \to H^i(P, \mathcal{F}(n)) \to H^{i+1}(P, \mathcal{R}(n)) \to \cdots$$

From the description given in *(1)*, it is clear that $H^i(P, \mathcal{E}(n)) = 0$ for large enough n and when $i > 0$ and $H^{i+1}(P, \mathcal{R}(n)) = 0$ from the induction hypothesis. Thus, $H^i(P, \mathcal{F}(n)) = 0$. Thus, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_O$, $H^n(P, \mathcal{F}) = 0$ □

3.4 Ext Groups and Sheaves

Definition. Let X be any ringed space and $\mathcal{F}$ a sheaf of modules on X. Then, we define $\mathsf{Ext}^i(\mathcal{F}, \cdot)$ for $i \geq 0$ to be the right derived functor of the functor $\mathsf{Hom}(\mathcal{F}, \cdot)$ from $\mathscr{Mod}(X)$ to **Ab**. We define the functor $\mathscr{Ext}^i(\mathcal{F}, \cdot)$ for $i \geq 0$ to be the right derived functor of the functor $\mathscr{Hom}(\mathcal{F}, \cdot)$ from $\mathscr{Mod}(X)$ to itself.

Proposition 3.14. *For any open subset* $U \subseteq X$, $\mathscr{Ext}^i_X(\mathcal{F}, \mathcal{G})|_U \cong \mathscr{Ext}^i_U(\mathcal{F}|_U, \mathcal{G}|_U)$

Proof. [3, page 234] □

Proposition 3.15. *Let* X *be a ringed space and* $\mathcal{F} \in \mathscr{Mod}(X)$. *Then,* $\mathsf{Ext}^i(\mathcal{O}_X, \mathcal{F}) \cong H^i(X, \mathcal{F})$ *for all* $i \geq 0$.

Proof. This follows immediately form the fact that $\mathscr{Hom}(\mathcal{O}_X, \mathcal{F}) \cong \mathcal{F}$. Thus $\mathsf{Hom}(\mathcal{O}_X, \mathcal{F}) \cong \Gamma(X, \mathcal{F})$. □

Proposition 3.16. *Let* $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ *be a short exact sequence in* $\mathscr{Mod}(X)$, *then for any* $\mathcal{G}$ *we have a long exact sequence* $0 \to \mathsf{Hom}(\mathcal{F}'', \mathcal{G}) \to \mathsf{Hom}(\mathcal{F}, \mathcal{G}) \to \mathsf{Hom}(\mathcal{F}', \mathcal{G}) \to \mathsf{Ext}^1(\mathcal{F}'', \mathcal{G}) \to \mathsf{Ext}^1(\mathcal{F}, \mathcal{G}) \to \mathsf{Ext}^1(\mathcal{F}', \mathcal{G}) \to \cdots$

Proof. [3, page 234] □

Proposition 3.17. *Let* $X = \mathsf{Spec}\ A$ *be an noetherian affine scheme and let* M *and* N *be* A-*modules with the added condition that* M *is finitely generated. Then, there is an isomorphism* $\mathscr{Ext}^i_X(\widetilde{M}, \widetilde{N}) \xrightarrow{\sim} \widetilde{\mathsf{Ext}^i_A(M, N)}$

Proof. Consider the following δ-functors. $\widetilde{\mathrm{Ext}^i_A(M,\cdot)}$ and $\mathscr{Ext}^i_X(\widetilde{M},\widetilde{\cdot})$ from the category of A-modules to the category of $\mathcal{O}_X$-modules. Now, $\widetilde{\mathrm{Ext}^i_A(M,\cdot)}$ is universal and we also have a natural morphism $\widetilde{\mathrm{Ext}^0_A(M,\cdot)} \cong \widetilde{\mathrm{Hom}_A(M,\cdot)} \to \mathscr{Hom}_X(\widetilde{M},\widetilde{\cdot}) \cong \mathscr{Ext}^0_X(\widetilde{M},\widetilde{\cdot})$. Since A is a noetherian and M is finitely generated, $\mathscr{Ext}^0_X(\widetilde{M},\widetilde{\cdot})$ is also a universal δ-functor and the above morphism is an isomorphism. Thus, $\mathscr{Ext}^i_X(\widetilde{M},\widetilde{\cdot}) \xrightarrow{\sim} \widetilde{\mathrm{Ext}^i_A(M,\cdot)}$ $\square$

Proposition 3.18. *Let X be a noetherian scheme and $\mathcal{F},\mathcal{G}$ be coherent sheaves on X. Then, $\mathscr{Ext}^i(\mathcal{F},\mathcal{G})$ is coherent for all $i \geq 0$*

Proof. Let $U = \mathrm{Spec}\, A$ be an affine open subset of X. Now, suppose $\mathcal{F}|_U = \widetilde{M}$ and $\mathcal{G}|_U = \widetilde{N}$. Then, $\mathscr{Ext}^i_X(\mathcal{F},\mathcal{G})|_U \cong \mathscr{Ext}^i_U(\widetilde{M},\widetilde{N}) \cong \widetilde{\mathrm{Ext}^i_A(M,N)}$. Since M and N are finitely generated A-modules, $\mathrm{Ext}^i_A(M,N)$ is finitely generated. Thus, $\mathscr{Ext}^i_X(\mathcal{F},\mathcal{G})$ is coherent. $\square$

Proposition 3.19. *Let $\mathcal{L}$ be a locally free sheaf of finite rank and let $\check{\mathcal{F}}$ be its dual. Then for any $\mathcal{F},\mathcal{G} \in \mathscr{Mod}(X)$ we have $\mathrm{Ext}^i(\mathcal{F} \bigotimes \mathcal{L},\mathcal{G}) \cong \mathrm{Ext}^i(\mathcal{F},\check{\mathcal{L}} \bigotimes \mathcal{G})$ and $\mathscr{Ext}^i(\mathcal{F} \bigotimes \mathcal{L},\mathcal{G}) \cong \mathscr{Ext}(\mathcal{F},\check{\mathcal{L}} \bigotimes \mathcal{G}) \cong \mathscr{Ext}^i(\mathcal{F},\mathcal{G}) \bigotimes \check{\mathcal{L}}$*

Proof. [3, page 235] $\square$

Proposition 3.20. *Let X be a noetherian scheme, let $\mathcal{F}$ be a coherent sheaf on X, let $\mathcal{G}$ be any $\mathcal{O}_X$-module, and let $x \in X$ be a point. Then we have $\mathscr{Ext}^i(\mathcal{F},\mathcal{G})_x \cong \mathrm{Ext}^i(\mathcal{F}_x,\mathcal{G}_x)$*

Proof. [3, page 235, 236] $\square$

Proposition 3.21. *Let X be a projective scheme over a noetherian ring A, let $\mathcal{O}_X(1)$ be a very ample invertible sheaf, and let $\mathcal{F},\mathcal{G}$ be coherent sheaves on X. Let $i \in \mathbb{N} \cup \{0\}$, then there is an integer $n_0 > 0$ (depending on $\mathcal{F},\mathcal{G}$ and i), such that for ecery $n \geq n_0$ we have $\mathrm{Ext}^i(\mathcal{F},\mathcal{G}(n)) \cong \Gamma(X,\mathscr{Ext}^i(\mathcal{F},\mathcal{G}(n)))$.*

Proof. [3, page 236] $\square$

Chapter 4

Serre duality Theorem

In this chapter k will represent a field and for any closed subscheme $X \subseteq \mathbb{P}^n_k$, the map $j : X \to \mathbb{P}^n_k$ will represent the natural inclusion map. Certain proofs in this section were referred from [5].

4.1 Serre Duality Theorem

Theorem 1. Duality for $\mathbb{P}^n_k$. *Let* $X = \mathbb{P}^n_k$ *and* ω_X *its canonical sheaf. Then:*

1. $H^n(X, \omega_X) \cong k$. *Fix one isomorphism;*

2. *For any coherent sheaf* $\mathcal{F}$ *on* X*, the natural pairing*

$$\mathrm{Hom}(\mathcal{F}, \omega_X) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X) \cong k$$

is a perfect paring of finitely generated vector spaces over k*;*

3. *for every* $i \geq 0$ *there is a natural functorial isomorphism*

$$\mathrm{Ext}^i(\mathcal{F}, \omega) \tilde{\to} H^{n-i}(X, \mathcal{F})^*$$

.

Proof. (1) Now, we know that if $X = \mathbb{P}^n_k$, then $\omega_X = \bigwedge^n \Omega_{X/k} \cong \mathcal{O}_X(-n-1)$. Thus, $H^n(X, \omega_X) \cong H^n(X, \mathcal{O}_X(-n-1)) \cong k$.
(2) For any homomorphism $\phi : \mathcal{F} \to \omega_X$, we have a natural homorphism $H^n(\phi) : H^n(X, \mathcal{F}) \to H^n(X, \omega_X)$. This gives us the desired pairing (bilinear map). We first prove the result for $\mathcal{F} = \mathcal{O}_X(q)$ for some $q \in \mathbb{Z}$. Now, $\mathrm{Hom}(\mathcal{F}, \omega_X) \cong \mathrm{Hom}(\mathcal{O}_X(q), \omega_X) \cong \mathrm{Hom}(\mathcal{O}_X, \check{\mathcal{O}_X}(q) \bigotimes \omega_X) \cong \mathrm{Hom}(\mathcal{O}_X, \mathcal{O}_X(-q) \bigotimes \omega_X) \cong \mathrm{Hom}(\mathcal{O}_X, \omega_X(-q)) \cong \Gamma(X, \omega_X(-q)) \cong H^0(X, \omega_X(-q)) \cong H^0(X, \mathcal{O}_X(-q-n-1))$. Therefore we have the natural pairing $H^0(X, \mathcal{O}_X(m)) \times H^n(X, \mathcal{O}_X(-m-n-1)) \to H^n(X, \mathcal{O}_X(-n-1)) \cong k$ where $m = -q-n-1$, which we know is a perfect pairing. Thus, if $\mathcal{F} = \bigoplus \mathcal{O}_X(q_i)$ is a finite direct sum, then since $\mathrm{Hom}(\mathcal{F}, \omega_X) \cong \bigoplus \mathrm{Hom}(\mathcal{O}_X(q_i), \omega_X)$ and $H^n(X, \mathcal{F}) \cong \bigoplus H^n(X, \mathcal{O}_X(q_i))$, the result is still true. Now, suppose that $\mathcal{F}$ is some coherent $\mathcal{O}_X$-module. Then, there exists a sheaf $\mathcal{E}_1$ which is a finite direct sum of sheaves of the form $\mathcal{O}_X(q_i)$ such that $\mathcal{E}_1 \to \mathcal{F} \to 0$ is exact. Now, since the kernel of the surjective map is also coherent, there exists a similar sheaf $\mathcal{E}_0$ such that the sequence $\mathcal{E}_0 \to \mathcal{E}_1 \to \mathcal{F} \to 0$ is exact. Now, $\mathrm{Hom}(\cdot, \omega_X)$ and $H^n(X, \cdot)^*$ are both left exact contravariant functors. Thus, by a

special case of the 5-lemma (by applying the functors on the sequence $\mathcal{E}_0 \to \mathcal{E}_1 \to \mathcal{F} \to 0 \to 0$) we get an isomorphism $\mathrm{Hom}(\mathcal{F}, \omega_X) \tilde{\to} H^n(X, \mathcal{F})^*$.
(3) Both $\mathrm{Ext}^i(\cdot, \omega_X)$ and $H^{n-i}(X, \cdot)^*$ are contravariant δ-functors from the category $\mathscr{Coh}(X)$ of coherent sheaves on X to **Ab**. Now, for $i = 0$, we have an isomorphism from $\mathrm{Ext}^0(\cdot, \omega_X) \cong \mathrm{Hom}(\cdot, \omega_X)$ to $H^n(X, \cdot)^*$. Thus, if we show that that both these functors are coeffaceable then they will both be universal and we will have morphisms form $\mathrm{Ext}^i(\cdot, \omega_X)$ to $H^{n-i}(X, \cdot)$ and the other way around whose composition will have to be identity because it is identity on when $i = 0$ and therefore we will be done. Suppose $\mathcal{F}$ is a coherent sheaf. Then, for some large enough $q \in \mathbb{N}$, $\mathcal{F}$ is the quotien of some finite direct sum $\mathcal{E} = \bigoplus \mathcal{O}_X(-q)$. Then, $\mathrm{Ext}^i(\mathcal{E}, \omega_X) \cong \mathrm{Ext}^i(\mathcal{O}_X, \omega_X \bigotimes (\bigoplus \mathcal{O}_X(q))) \cong \mathrm{Ext}^i(\mathcal{O}_X, \bigoplus(\omega_X(q))) \cong H^i(X, \bigoplus(\omega_X(q))) \cong \bigoplus H^i(X, \omega_X(q))$. Now, $\omega_X(q) \cong \mathcal{O}_X(-n-1)(q) \cong \mathcal{O}_X(-n-1+q)$. Thus, $H^i(X, \omega_X(q)) = 0$ for a large enough q. Therefore, $\mathrm{Ext}^i(\mathcal{E}, \omega_X) = 0$. Therefore the epimorphisms $e : \mathcal{E} \to \mathcal{F}$ goes to the zero morphism $\mathrm{Ext}^i(\mathcal{F}, \omega_X) \to \mathrm{Ext}^i(\mathcal{E}, \omega_X) = 0$. Therefore, $\mathrm{Ext}^i(\cdot, \omega_X)$ is universal. Similarly, $H^{n-i}(X, \mathcal{E})^* = \bigoplus H^{n-i}(X, \mathcal{O}_X(-q))^* = 0$ because $H^{n-i}(X, \mathcal{O}_X(-q)) = 0$. Thus, even $H^{n-i}(X, \cdot)^*$ is a universal functor. □

Definition. Let X be a projective scheme of dimension n over a field k. A $\mathsf{dualizing\ sheaf}$ for X is a coherent sheaf ω_X° on X together with a trace morphism $t : H^n(X, \omega_X^\circ) \to k$, such that for all coherent sheaves $\mathcal{F}$ on X, the natural pairing

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \times H^n(X, \mathcal{F}) \to H^n(X, \omega_X^\circ) \xrightarrow{t} k$$

gives an isomorphism

$$\mathrm{Hom}(\mathcal{F}, \omega_X^\circ) \tilde{\to} H^n(X, \mathcal{F})^*.$$

We can in fact define the dualizing sheaf for a type of schemes known as proper schemes but since we are interested in projective schemes, I have defined it only for those (all projective morphisms are proper).

Proposition 4.1. *Let X be a projective scheme over k. Then, a dualizing sheaf (ω_X°, t) for X, if it exists, is unique up to isomorphism.*

Proof. [3, page 241] □

Proposition 4.2. *Let X be a closed subscheme of codimension r of $P = \mathbb{P}^N_k$. Then, $\mathscr{Ext}^i(j_*\mathcal{O}_X, \omega_P) = 0$ for all $i < r$.*

Proof. Let us denote by $\mathcal{F}^i = \mathscr{Ext}^i(j_*\mathcal{O}_X, \omega_P)$. Since $j_*\mathcal{O}_X$ and ω_P are coherent, $\mathcal{F}^i$ is coherent for each $i \geq 0$. Therefore there exists a $q_i > 0$ such that for all $q \geq q_i$, $\mathcal{F}^i(q)$ are generated by global sections. Now, to show that $\mathcal{F}^i = 0$, it will be sufficient to show that $\mathcal{F}^i(q_i) = 0$ (because $\mathcal{F}^i(q_i)$ and $\mathcal{F}^i$ have isomorphic stalks). But, $\mathcal{F}^i(q_i) = 0$ if and only if for all $x \in P$, $\mathcal{F}^i(q_i)_x = 0$, Since $\mathcal{F}^i(q_i)$ is generated by global section, if we show $\Gamma(P, \mathcal{F}^i(q_i)) = 0$, then for all $x \in P$, $\mathcal{F}^i(q_i)_x = 0$. We know that $\Gamma(P, \mathcal{F}^i(q_i)) = \Gamma(P, \mathscr{Ext}^i(j_*\mathcal{O}_X, \omega_P) \bigotimes \mathcal{O}_P(q_i)) \cong \Gamma(P, \mathscr{Ext}^i(j_*\mathcal{O}_X, \omega_P(q_i))) \cong \mathrm{Ext}^i_P(j_*\mathcal{O}_X, \omega_P(q_i)) \cong \mathrm{Ext}^i_P(j_*\mathcal{O}_X(-q_i), \omega_P) \cong H^{N-i}(P, j_*\mathcal{O}_X(-q))^*$. Now, $H^{N-i}(P, j_*\mathcal{O}_X(-q_i)) \cong H^{N-i}(X, \mathcal{O}_X(-q_i))$ and if $i < r$, then $N - i > N - r = \dim X$. Therefore, by Grothendieck Vanishing theorem, $H^{N-i}(P, j_*\mathcal{O}_X(-q_i)) = 0$. Therefore, $\Gamma(P, \mathcal{F}^i(q_i)) \cong H^{N-i}(P, j_*\mathcal{O}_X(-q_i))^* = 0$. This completes the proof. □

Proposition 4.3. *With the same premise as the above theorem, let $\omega_X^\circ = j^* \mathscr{Ext}^r_P(j_*\mathcal{O}_X, \omega_P)$. Then, for any $\mathcal{O}_X$-module $\mathcal{F}$, there is a functorial isomorphism*

$$\mathrm{Hom}_X(\mathcal{F}, \omega_X^\circ) \cong \mathrm{Ext}^r_P(j_*\mathcal{F}, \omega_P)$$

Proof. Let $0 \to \omega_P \to \mathscr{I}^\bullet$ be an injecitve resolution of ω_P in $\mathscr{Mod}(P)$. Denote by $\mathcal{J}^i = \mathscr{Hom}_P(j_*\mathcal{O}_X, \mathscr{I}^i)$. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then, any morphism $j_*\mathcal{F} \to \mathscr{I}^i$ will naturally factor through $\mathcal{J}^i$ and so it is easy to see that $\mathrm{Hom}_X(\mathcal{F}, j^*\mathcal{J}^i) \cong \mathrm{Hom}_P(j_*\mathcal{F}, \mathscr{I}^i)$. Thus, we have that $\mathrm{Ext}^i(\mathcal{F}, \omega_P) \cong h^i(\mathrm{Hom}_X(\mathcal{F}, j^*\mathcal{J}^\bullet))$ and we also prove that $\mathrm{Hom}_X(\cdot, j^*\mathcal{J}^i)$ is exact. Now, $h^i(\mathcal{J}^\bullet) = \mathscr{Ext}^i_P(j_*\mathcal{O}_X, \omega_P)$. Thus, $h^i(\mathcal{J}^\bullet) = 0$ for all $i < r$. Let $K^{i-1} \subseteq \mathcal{J}^i$ be the image subsheaf of the morphism $\mathcal{J}^{i-1} \to \mathcal{J}^i$ for $1 < i \leq r$. Then, we have the following r exact sequences.

$$0 \to \mathcal{J}^0 \to \mathcal{J}^1 \to K^1 \to 0$$
$$0 \to K^1 \to \mathcal{J}^2 \to K^2 \to 0$$
$$\vdots$$
$$0 \to K^{r-2} \to \mathcal{J}^{r-1} \to K^{r-1} \to 0$$
$$0 \to K^{r-1} \to \mathcal{J}^r \to \mathcal{J}^r/K^{r-1} \to 0$$

Now, $\mathcal{J}_0$ is injective as we have seen above. Thus, the sequence splits and therefore K^1 is injective. By folowing this process we get that $0 \to K^{r-1} \to \mathcal{J}^r \to \mathcal{J}^r/K^{r-1} \to 0$ splits and therefore $\mathcal{J}^r \cong K^{r-1} \bigoplus \mathcal{J}^r/K^{r-1}$. Since K^{r-1} is contained in the kernel of the morphism $\mathcal{J}^r \to \mathcal{J}^{r+1}$, we get a natural morphism $\mathcal{J}^r/K^{r-1} \to \mathcal{J}^{r+1}$. Let us denote by K, the kernel of this morphism. Now, we get $\mathscr{Ext}^r_P(j_*\mathcal{O}_X, \omega_P)) \cong (K^{r-1} \oplus K)/K^{r-1} \cong K$. Using this result, if we define

$f : \mathrm{Hom}_X(\mathcal{F}, j^*\mathcal{J}^r) \to \mathrm{Hom}_X(\mathcal{F}, j^*\mathcal{J}^{r+1})$ and let $g : \mathrm{Hom}_X(\mathcal{F}, j^*\mathcal{J}^{r-1}) \to \mathrm{Hom}_X(\mathcal{F}, j^*\mathcal{J}^r)$, then $\ker f = \mathrm{Hom}_X(\mathcal{F}, j^*(K^{r-1} \bigoplus K))$ and $\mathrm{img}\ g = \mathrm{Hom}_X(\mathcal{F}, j^*K^{r-1})$, therefore
$h^i(\mathrm{Hom}_X(\mathcal{F}, j^*\mathcal{J}^\bullet)) = \ker f/\mathrm{img}\ g \cong \mathrm{Hom}_X(\mathcal{F}, j^*K) \cong \mathrm{Hom}_X(\mathcal{F}, j^* \mathscr{Ext}^r_P(j_*\mathcal{O}_X, \omega_P))) \cong$ $\mathrm{Hom}_X(\mathcal{F}, \omega^\circ_P)$. Thus, we have the desired result that $\mathrm{Hom}_X(\mathcal{F}, \omega^\circ_X) \cong \mathrm{Ext}^r_P(j_*\mathcal{F}, \omega_P)$. □

Proposition 4.4. *Let X be a projective scheme over k. Then X has a dualizing sheaf.*

Proof. Let X and ω°_X be as defined in the above proposition. Now, since $\mathscr{Ext}^r_P(j_*\mathcal{O}_X, \omega_P)$ is a coherent sheaf over P, we have that ω°_X is a coherent sheaf over X. We know that $\mathrm{Hom}_X(\mathcal{F}, \omega^\circ_X) \cong \mathrm{Ext}^r_P(j_*\mathcal{F}, \omega_P) \cong H^{N-r}(P, j_*\mathcal{F})^* \cong H^{N-r}(X, \mathcal{F})^* = H^n(X, \mathcal{F})^*$, where n is the dimension of X. Now, we have the desired isomorphism $\mathrm{Hom}_X(\mathcal{F}, \omega^\circ_X) \cong H^n(X, \mathcal{F})^*$ As a special case we have $\mathrm{Hom}_X(\omega^\circ_X, \omega^\circ_X) \cong H^n(X, \omega^\circ_X)^*$. Thus the identity morphism in $\mathrm{Hom}_X(\omega^\circ_X, \omega^\circ_X)$ corresponds to a homomorphism $t : H^n(X, \omega^\circ_X) \to k$. Then (ω°_X, t) is the required dualizing sheaf which gives the above above proved isomorphism. □

Theorem 2. Duality for a Projective Scheme. *Let X be a projective scheme of dimension n over an algebraically closed field k. Let ω°_X be a dualizing sheaf on X, and let $\mathcal{O}(1)$ be a very ample sheaf on X. Then:*

1. *For all $i \geq 0$ and $\mathcal{F}$ coherent one X, there are natural functorial maps*

$$\theta^i : \mathrm{Ext}^i(\mathcal{F}, \omega^\circ_X) \to H^{n-i}(X, \mathcal{F})^*$$

   such that θ^0 is the existing map $\mathrm{Ext}^0(\mathcal{F}, \omega^\circ_X) \cong \mathrm{Hom}_X(\mathcal{F}, \omega^\circ_X) \to H^n(X, \mathcal{F})^*$

2. *The following conditions are equivalent:*

   (a) *X is Cohen-Macaulay and equidimensional.*

   (b) *For any $\mathcal{F}$ locally free on X, we have $H^i(X, \mathcal{F}(-q)) = 0$ for all $i < n$ and all large enough q.*

(c) The maps θ^i *from (1) are isomorphims for all* $i \geq 0$ *and all* $\mathcal{F}$ *coherent on* X.

Proof. (1) We prove this in a similar manner as the last theorem. We know that $\mathcal{F}$ is the quotient of a finite direct sum $\mathcal{E} = \bigoplus_{i=1}^N \mathcal{O}_X(-q)$ for all large enough $q > 0$. Now, $\mathrm{Ext}^i(\mathcal{E}, \omega_X^\circ) \cong \bigoplus H^i(X, \omega_X^\circ(q))$ in the same manner are before. Now, for all $i > 0$ and all large enough q we know that $H^i(X, \omega_X^\circ(q)) = 0$ since ω_X° is coherent. Therefore, $\mathrm{Ext}^i(\cdot, \omega_X^\circ)$ is coeffaceable for $i > 0$ and thus universal. Now, there exists a map $\theta^0 : \mathrm{Ext}^0(\mathcal{F}, \omega_X^\circ) \cong \mathrm{Hom}_X(\mathcal{F}, \omega_X^\circ) \to H^n(X, \mathcal{F})^*$. Therefore, for each $i \geq 0$ we have a morphism $\theta^i : \mathrm{Ext}^i(\mathcal{F}, \omega_X^\circ) \to H^{n-i}(X, \mathcal{F})^*$.
(2)(a) $\implies$ *(b)* Let $\mathcal{F}$ be a locally free sheaf on X and $x \in X$ be a closed point. Now, since X is Cohen Macaulay and equidimensional and $\mathcal{F}_x$ is a free $\mathcal{O}_{X,x}$-module, therefore $\mathrm{depth}\ \mathcal{F}_x = \mathrm{depth}\ \mathcal{O}_{X,x} = \dim \mathcal{O}_{X,x} = \dim X = n$. Let $A = \mathcal{O}_{P,x}$. Since P is a nonsingular vatiety, A is regular. Moreover, $\mathrm{depth}\ \mathcal{F}_x$ is independent of wether the base ring is A or $\mathcal{O}_{X,x}$. Therefore , $\mathrm{pd}_A \mathcal{F}_X + \mathrm{depth}_A \mathcal{F}_x = N$. Thus, we have $\mathrm{pd}_A \mathcal{F}_x = N - n$. Now, since $j_* \mathcal{F}$ is an extension of $\mathcal{F}$ by 0 outside X, $(j_*\mathcal{F})_x = \mathcal{F}_x$, and since $\mathrm{pd}_A (j_*\mathcal{F})_x \leq N - n$, $\mathrm{Ext}_A^i((j_*\mathcal{F}), N) = 0$ for all A-modules N and all $i > N - n$. Now, for all $x \in P$, $\mathscr{E}xt_P^i(j_*\mathcal{F}, \mathcal{G})_x \cong \mathrm{Ext}_A^i((j_*\mathcal{F})_x, \mathcal{G}_x)$. Now, if $x \notin X$, then this is zero because $(j_*\mathcal{F})_x = 0$ and if $x \in X$, then we know it is zero whenever $i > N - n$. Thus, $\mathscr{E}xt_P^i(j_*\mathcal{F}, \mathcal{G})_x = 0$ for all $x \in X$ and $i > N - n$. Thus, $\mathscr{E}xt_P^i(j_*\mathcal{F}, \cdot) = 0$ whenever $i > N - n$. Now, we know that $H^i(X, \mathcal{F}(-q))^* \cong H^i(P, j_*\mathcal{F}(-q))^* \cong \mathrm{Ext}_P^{N-i}(j_*\mathcal{F}, \omega_P(q))$. For all large enough q, we have $\mathrm{Ext}_P^{N-i}(j_*\mathcal{F}, \omega_P(q)) \cong \Gamma(P, \mathscr{E}xt_P^{N-i}(j_*\mathcal{F}, \omega_P(q)))$. We have just shown that if $N - i > N - n$, then $\Gamma(P, \mathscr{E}xt_P^{N-i}(j_*\mathcal{F}, \omega_P(q))) = 0$. Thus for all large enough q and if $i < n$, then $H^i(X, \mathcal{F}(-q)) = 0$.
(b) $\implies$ *(a)* First we wish to show that $\mathscr{E}xt_P^i(j_*\mathcal{O}_X, \omega_P) = 0$ for all $i > N - n$. To show this it would suffice to show that $\mathscr{E}xt_P^i(j_*\mathcal{O}_X, \omega_P)(q) \cong \mathscr{E}xt_P^i(j_*\mathcal{O}_X, \omega_P(q)) = 0$ for some q. Since $\mathscr{E}xt_P^i(j_*\mathcal{O}_X, \omega_P)$ is coherent, we know that $\mathscr{E}xt_P^i(j_*\mathcal{O}_X, \omega_P(q))$ is generated by global section for a large enough q. We have already shown that $\Gamma(P, \mathscr{E}xt_P^i(j_*\mathcal{O}_X, \omega_P(q))) \cong H^{N-i}(X, \mathcal{O}_X(-q))^*$. Now, if $i > N - n$, then $N - i < n$. Thus, $H^{N-i}(X, \mathcal{O}_X(-q)) = 0$ and thus we have $\Gamma(P, \mathscr{E}xt_P^i(j_*\mathcal{O}_X, \omega_P(q))) = 0$. Therefore, $\mathscr{E}xt_P^i(j_*\mathcal{O}_X, \omega_P) = 0$ for all $i > N - n$. Thus, for any closed point $x \in X$ and local ring $A = \mathcal{O}_{P,x}$, we have that $\mathrm{Ext}_A^i((j_*\mathcal{O}_X)_x, (\omega_P)_x) \cong \mathrm{Ext}_A^i((j_*\mathcal{O}_X)_x, A) = 0$ for all $i > N - n$ and thus we have that $\mathrm{pd}_A (j_*\mathcal{O}_X)_x \leq N - n$ and therefore $\mathrm{depth}\ \mathcal{O}_{X,x} \geq n$ because $\mathcal{O}_{X,x} \cong (j_*\mathcal{O}_X)_x$. But, we know that $\mathrm{depth}\ \mathcal{O}_{X,x} \leq \dim \mathcal{O}_{X,x} \leq n$. Thus, $\mathrm{depth}\ \mathcal{O}_{X,x} = \dim \mathcal{O}_{X,x} = n$. Thus, $\mathcal{O}_{X,x}$ is Cohen Macaulay for all closed point $x \in X$ Since every other point is basically the localisation of the local ring of a closed point at some prime ideal it is also Cohen Macaulay. Thus, the scheme is Cohen-Macaulay. It is equidimensional because the dimension of all closed points are equal.
(b) $\implies$ *(c)* As in the proof of Duality for Projective spaces, it will suffice to show that $H^{n-i}(X, \cdot)^*$ is coeffaceable for $i > 0$ and hence universal. Let $\mathcal{F}$ be a coherent sheaf. Then, we can write it as a quotient of a sheaf of the form $\mathcal{E} = \bigoplus_{i=1}^N \mathcal{O}_X(-q)$ for a large enough $q > 0$. Now, $H^{n-i}(X, \mathcal{O}_X(-q)) = 0$ for all $i > 0$ because $\mathcal{O}_X(-q)$ is locally free and hence we have $H^{n-i}(X, \mathcal{E}) = H^{n-i}(X, \mathcal{E})^* = 0$ there for all $i > 0$.
(c) $\implies$ *(b)* Let $\mathcal{F}$ be a locally free sheaf on X. Since $H^i(X, \mathcal{F}(-q))$ a finitely generated k-vector, we have $H^i(X, \mathcal{F}(-q)) \cong H^i(X, \mathcal{F}(-q))^{**} \cong \mathrm{Ext}^{n-i}(\mathcal{F}(-q), \omega_X^\circ)^* \cong \mathrm{Ext}^{n-i}(\mathcal{F}, \omega_X^\circ(q))^* \cong \mathrm{Ext}^{n-i}(\mathcal{O}_X, \check{\mathcal{F}} \bigotimes \omega_X^\circ(q))^* \cong H^{n-i}(X, \check{\mathcal{F}} \bigotimes \omega_X^\circ(q))^*$ whenever $n - i > 0$. We know that for all large enough q and when $n - i > 0$, $H^{n-i}(X, \check{\mathcal{F}} \bigotimes \omega_X^\circ(q)) = 0$. Thus, when $i < n$ for all large enough q, $H^i(X, \mathcal{F}(-q)) = 0$. ◻

4.2 Kodaira Vanishing Theorem

Before we discuss the two equivalent versions of the Kodaira Vanishing Theorem, we need to prove a corollary of the Serre Duality Theorem.

Proposition 4.5. *Let X be a projective Cohen-Macaulay scheme of equidimension n over k. Then for any locally free sheaf $\mathcal{F}$ on X and any $0 \le i \le n$, there are natrual isomorphisms:*

$$H^i(X, \mathcal{F}) \cong H^{n-i}(X, \check{\mathcal{F}} \bigotimes \omega_X^\circ)^*$$

Proof. Since $n - i \ge 0$, we have $H^{n-i}(X, \check{\mathcal{F}} \bigotimes \omega_X^\circ) \cong \mathrm{Ext}^{n-i}(\mathcal{O}_X, \check{\mathcal{F}} \bigotimes \omega_X^\circ) \cong \mathrm{Ext}^{n-i}(\mathcal{F}, \omega_X^\circ)$. Now, again since $n - i \ge 0$, we have $\mathrm{Ext}^{n-i}(\mathcal{F}, \omega_X^\circ) \cong H^{n-(n-i)}(X, \mathcal{F})^* = H^i(X, \mathcal{F})^*$. Therefore, we have $H^{n-i}(X, \check{\mathcal{F}} \bigotimes \omega_X^\circ) \cong H^i(X, \mathcal{F})^*$. Thus, we have the desired result

$$H^{n-i}(X, \check{\mathcal{F}} \bigotimes \omega_X^\circ)^* \cong H^i(X, \mathcal{F})^{**} \cong H^i(X, \mathcal{F})$$

□

Theorem 3. Kodaira Vanishing Theorem. *If X is a projective non-sngular variety of dimension n over $\mathbb{C}$, and if $\mathcal{I}$ is an ample invertible sheaf on X, then $H^i(X, \mathcal{I} \bigotimes \omega_X^\circ) = 0$ for all $i > 0$.*

We shall not be proving the Kodaira Vanishing theorem as that is not intent of this thesis. Instead, using Serre Duality we shall be showing that given the same premise as Kodaira vanishing theorem, having $H^i(X, \mathcal{I} \bigotimes \omega_X^\circ) = 0$ for all $i > 0$ is equivalent to having $H^i(X, \mathcal{I}^{-1}) = 0$ for all $i < n$.

Proposition 4.6. *If X is a projective non-singular variety of dimension n over $\mathbb{C}$, and if $\mathcal{I}$ is an invertible sheaf on X, then the following statements are equivalent:*

1. $H^i(X, \mathcal{I} \bigotimes \omega_X^\circ) = 0$ *for all* $i > 0$*;*

2. $H^i(X, \mathcal{I}^{-1}) = 0$ *for all* $i < n$*.*

Proof. Now, if X is non-singular, then each of its local rings are noetherian local and regular. Since all regular noetherian local rings are Cohen Macaulay, X is Cohen Macaulay. Moreover, since X is a vairety, so it is irreducible and hence equidimensional. Now, since X is Cohen Macaulay and equidimensional, we know from the above proposition that for any invertible sheaf $\mathcal{I}$, we have $H^i(X, \mathcal{I}^{-1}) \cong H^i(X, \check{\mathcal{I}}) \cong H^{n-i}(X, (\check{\mathcal{I}})\check{} \bigotimes \omega_X^\circ)^* \cong H^{n-i}(X, \mathcal{I} \bigotimes \omega_X^\circ)^*$ for any $0 \le i \le n$. Now, we can prove the equivalence of the two statements.
(1) $\Longrightarrow$ (2) Suppose $H^i(X, \mathcal{I} \bigotimes \omega_X^\circ) = 0$ for all $i > 0$. Let $j = n - i$. Now, $H^{n-j}(X, \mathcal{I} \bigotimes \omega_X^\circ) = 0$ for all $j < n$. If, $j < 0$, then $H^j(X, \mathcal{I}^{-1}) = 0$. If $0 \le j < n$, Then $H^j(X, \mathcal{I}^{-1}) \cong H^{n-j}(X, \mathcal{I} \bigotimes \omega_X^\circ)^* = 0$. Thus, $H^j(X, \mathcal{I}^{-1}) = 0$ for all $j < n$.
(2) $\Longrightarrow$ (1) Suppose $H^i(X, \mathcal{I}^{-1}) = 0$ for all $i < n$. In a similar manner as above let $j = n - i$. Now, if $i < 0$, then $j = n - i > n$. Thus, by Grothendieck Vanishing Theorem $H^j(X, \mathcal{I} \bigotimes \omega_X^\circ) = 0$. If, $0 < j \le n$ (i.e. $0 \le i < n$), we have $H^j(X, \mathcal{I} \bigotimes \omega_X^\circ)^* \cong H^i(X, \mathcal{I}^{-1}) = 0$. Thus, we have $H^j(X, \mathcal{I} \bigotimes \omega_X^\circ) = 0$ for all $j > 0$. □

Note that in this proposition, we do not state or use ampleness unlike in the Kodaira Vanishing Theorem. Then equivalence of the two statements does not require $\mathcal{I}$ to be ample, but if $\mathcal{I}$ isn't ample, then $H^i(X, \mathcal{I} \bigotimes \omega_X^\circ)$ need not be zero for all $i > 0$, which is why ampleness in neccessary in the statement of Kodaira Vanishing Theorem.

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