Talk 5

BOREL-SERRE COMPACTIFICATION - I

§1. SYMMETRIC SPACES.

- Modular forms as defined classically can be thought of all "algebraic" differential forms on the space

$$\Gamma \backslash \mathcal{H}^2$$

for some level $\Gamma \leq SL_2(\mathbb{Z})$.

- This space $\Gamma \backslash \mathcal{H}^2$ is very interesting because they can be studied as moduli spaces of elliptic curves with different level structures.

For example: (all elliptic curves are over $\mathbb{C}$)

(i) $\Gamma = \Gamma_0(N)$: Pairs (E, C), where

E - elliptic curve

$C \subseteq E[N]$ cyclic subgroup of order N

(ii) $\Gamma = \Gamma_1(N)$: Pairs (E, Q), where

E - elliptic curve

$Q \in E[N]$ point of order N.

(iii) $\Gamma = \Gamma(N)$: Triple (E, P, Q), where

E - elliptic curve

$P, Q \in E[N]$ such that $e_N(P, Q) = e^{2\pi i/N}$.

Weil pairing.

Moreover, in the above examples, we can find a compact open subgroup $K \leq SL_2(\mathbb{A}_f)$ such that $\Gamma = K \cap SL_2(\mathbb{Q})$ and

$$\Gamma \backslash \mathcal{H}^2 \cong SL_2(\mathbb{Q}) \backslash \mathcal{H}^2 \times SL_2(\mathbb{A}_f) / K$$

This suggest that there is an action of $SL_2(\mathbb{A}_f)$ on the system $\{\Gamma \backslash \mathcal{H}^2\}_\Gamma$

$\boxed{\text{Here, the } \Gamma \text{ must be chosen carefully}}$.

And if one was able to construct a "rational" model of these, then one could also define a Galois action on these structures.

In some sense, these objects capture some automorphic and something Galois simultaneously making them very intersting.

<u>Question</u>: Can we construct more general spaces like these?

<u>Answer</u>: Shimura Varieties

Definition: A symmetric space is a connected Riemannian manifold M such that:

(i) (Homogenity) The automorphism group of M acts transitively on M.

(ii) (Symmetricity) for all $p \in M$, there exists an involution s_p of M such that p is an isolated fixed point of p.

Remark: Most symmetric spaces we care about are certain "nice" Riemannian manifold called "Hermatian symmetric domain" (negative curvature)

Lemma: Let $p \in M$ be a point in a symmetric space & $K_p \leq \mathrm{Isom}(M)^+$ be the subgroup fixing p. Then, K_p is compact and

$$\mathrm{Isom}(M)^+ / K_p \longrightarrow M$$

is an isomorphism. In particular $\mathrm{Iso}(M)^+$ acts transitively on M.

Proof: Refer to Milne - Lemma 1.5 ____ $\boxtimes$

<u>Theorem</u>: Suppose M is a hermatian symmetric domain and $\mathfrak{g}$ is the Lie algebra of $\mathrm{Isom}(M)^+$. Then, there is a unique connected algebraic subgroup G of $GL(\mathfrak{g})$ such that:

$$G(\mathbb{R})^+ = \mathrm{Isom}(M)^+ \qquad (\text{inside } GL(\mathfrak{g}))$$

and

$$G(\mathbb{R})^+ = \mathrm{Isom}(M) \cap G(\mathbb{R}) \quad (\text{inside } GL(\mathfrak{g}))$$

Suppose $G/\mathbb{Q}$ is a connected reductive group. Additionally, let $K \subseteq G(\mathbb{R})$ be a fixed maximal compact subgroup and $A_G \subseteq G$ a maximal $\mathbb{Q}$-split torus in the center of G, and $A_G = A_G(\mathbb{R})^+$. Then, we call

$$X = G(\mathbb{R}) / K A_G$$

a symmetric space associated to G.

If $\Gamma \leq G(\mathbb{Q})$ is an arithmetic subgroup, then we call

$$X_\Gamma = \Gamma \backslash X = \Gamma \backslash G(\mathbb{R}) / K A_G$$

the locally symmetric space associated to G.

<u>Examples</u>: ① Notice that $\mathcal{H}^2$ is the symmetric space associated to SL_2.

② For $G = Sp_{2g}$, we get the higher dimensional Siegel upper half planes

$$\mathcal{H}^{2g} := \{ Z \in M_{g \times g}(\mathbb{C}) : Z^t = Z \ \& \ \mathrm{Im}(Z) > 0 \}$$

which, for different levels Γ, classifies abelian varieties of dimension g with some level structure.

③ Suppose $G = \mathrm{Res}_{\mathbb{Q}[i]/\mathbb{Q}} SL_2$. Then, the corresponding symmetric spaces is $SL_2(\mathbb{C}) / SU_2(\mathbb{C}) \cong \mathcal{H}^3$.

The locally symmetric spaces are called Bianchi manifolds. This clearly does not have a complex structure. This implies that there are not hermitian, & also cannot have the structure of a complex variety.

To see the adelic prespective of these locally symmetric spaces, suppose $\mathcal{G}/\mathbb{Z}$ is an integral model of G. Let $K_f \subseteq G(\mathbb{A}_f)$ be a compact open subgroup of the form $K_f = \prod_v K_v$ where $K_v \subseteq \mathcal{G}(\mathbb{Z}_v)$ and equality holds for all but finitely many v. Firstly, $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/K$ is finite.

Let $\Gamma_i = G(\mathbb{Q}) \cap g_i K_f g_i^{-1}$ where $g_1, \dots, g_r$ are double coset representatives. Then,

$$X_{K_f} = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K_f = \bigsqcup_i X_{\Gamma_i}$$

As discussed in previous talks, these X_{Γ_i}'s can be non-compact. There exist different types of compactifications:

① Bailey-Borel (If you want a (potentially singular) variety)
② Toroidal (singularities resolved in BB)
③ Borel-Serre. (CW-complex)

And more!!

We shall denote X_Γ^{BS} the Borel-Serre compactification. Now, the inclusion

$$X_\Gamma \hookrightarrow X_\Gamma^{BS}$$

is infact a homotopy equivalence.

<u>Consequence</u>: $H^i_{\text{Betti}}(X_\Gamma, A)$ is a finitely generated A-module for all nice $A = \mathbb{C}, \mathbb{Z}/p^N\mathbb{Z}, \mathbb{Q}_p$.

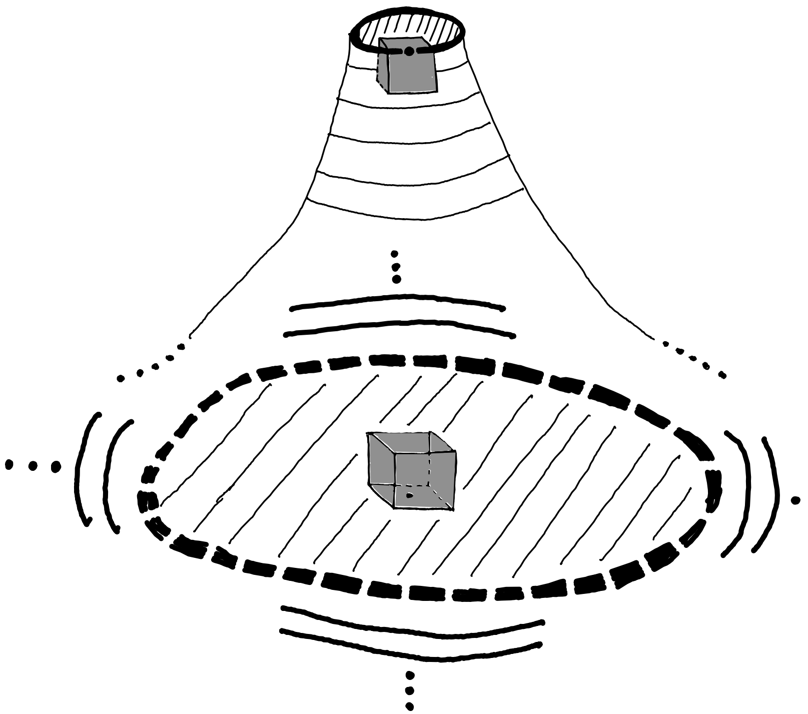
§2. THE IDEA

The Borel-Serre compactification is a (topologically) smooth manifold with a boundary. This boundary also has a differentiable structure.

Suppose X_Γ is manifold of dimension m.
Then, the boundary in X_Γ^{BS} is made up of smooth manifolds of various dimensions.

For instance if we consider a point in a boundary component of dimsion d. Then, there exists a neighbourhood of this point of the form $B \times [0,1)^{m-d}$ where B

is an open subset of $\mathbb{R}^d$.



This compactification is achieved by attaching to X boundery components associated to each proper parabolic subgroup of G giving us a partial compactification X^{BS}, and then quotienting it by Γ.

END OF TALK 5

Talk 6

Recall:

G: is a connected reductive group over $\mathbb{Q}$.

$A_G \subseteq G$: A maximal $\mathbb{Q}$-split torus in the identity component of the center of G.

$A_G = A_G(\mathbb{R})^+$

§3. THE GROUP ${}^\circ G$

Define:

$$\boxed{{}^\circ G := \bigcap_{\chi} \ker(\chi^2)}$$

where $\chi : G \longrightarrow \mathbb{G}_m$ runs over all rational characters of G.

The group ${}^\circ G$ is also a ~~connected~~ reductive algebraic group over $\mathbb{Q}$.

The group ${}^{\circ}G$ contains all compact open subgroups and all arithmetic subgroups of G.

Now,

$$G(\mathbb{R}) = {}^{\circ}G(\mathbb{R}) \times A_G$$

This is important, because if $K \leq G(\mathbb{R})$ as before is a maximal compact subgroup, then

$$X = G(\mathbb{R})/KA_G = {}^{\circ}G(\mathbb{R})/K$$

Because of the above fact, we shall assume that $A_G = 1$. This is not the best assumption in the content of Shimura varieties, but it helps with computations.

We shall make a few more assumptions to help simplify certain thing:

(i) Suppose G is such that X is Hermitian (has a G-invariant complex structure.

(ii) We shall assume Γ is torsion free so that the action $\Gamma \curvearrowright X$ is free. A stronger assumption of Γ being neat helps us show that the boundary strata are also smooth manifolds.

§4. GEODESIC ACTION

Firstly, for any parabolic subgroup $\mathbb{P} \subseteq G$, we denote by $P := \mathbb{P}(\mathbb{R})$.

If the Levi decomposition is

$$\mathbb{P} = \mathbb{U}_{\mathbb{P}} \rtimes \mathbb{L}_{\mathbb{P}}$$

And, we denote $L_P = \mathbb{L}_P(\mathbb{R})$ & $U_P = \mathbb{U}_{\mathbb{P}}(\mathbb{R})$

Then, if $M_P = {}^0\mathbb{L}_P(\mathbb{R})$ and A_P are the real points of the maximal split torus in the center of $\mathbb{L}_P$. Then the Langlands decomposition.

$$P = U_P A_P M_P.$$

(*) Firstly, it can be checked that $K_P = K \cap P$ is completely contained in M_P.

(*) Secondly, the Iwasawa decomposition tells us that P acts transitively on X.

So,

$$X = P/K_p.$$

And, we can define a right action of A_p on X in the following manner. (this action is called the geodesic action).

Suppose $gK_p \in X$ and $a \in A_p$

Then $(gK_p)\cdot a := gaK_p$

One can also verify this action is base point independent.

$$\begin{aligned} P/K_p &\longrightarrow X \\ IK_p &\longmapsto \underset{\substack{\uparrow \\ \text{base point}}}{x} \end{aligned}$$

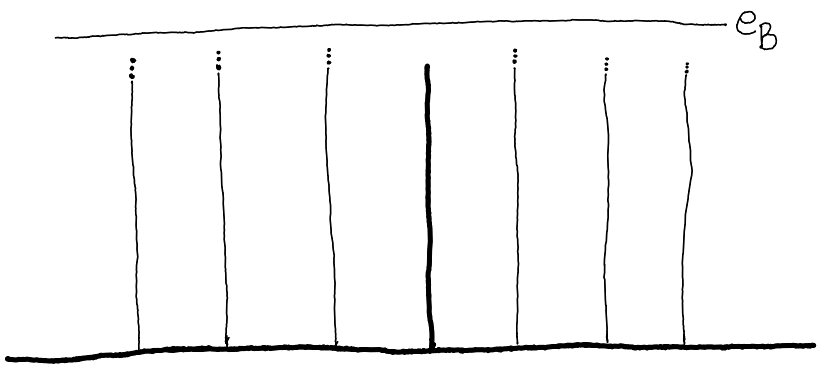
Important Fact: each orbit of the action of A_p is a totally geodesic submanifold of X. We define the (Borel-Serre) boundary component.

$$e_p = D/A_p$$

Example: Suppose $G = SL_2$ and $B \subseteq G(\mathbb{R})$ the upper triangular matrices. Then, $X = \mathcal{H}^2$ and A_B are the set of elements of the form $a = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$.

And, the action gives us $(x+iy)\cdot a = x + it^2 y$.

So, each element of e_B is a vertical line in $\mathcal{H}^2$. And so, we can represent e_B as a line at ∞



The idea when constructing the partial compactification is to attach these different

e_P to the boundary of X.

In the example of $G = SL_2$, we will have

$$\mathcal{H} \cup e_B = \mathbb{R} \times (0, \infty]$$

where the $\mathbb{R} \times \{\infty\}$ is exactly e_B.

Back to the general case:

We want Γ to act on $X \cup e_P$ in such way that $\Gamma_P \backslash e_P$, where $\Gamma_P = \Gamma \cap P$ attaches to the boundary of X_Γ.

Problem: Γ doesn't really act on $X \cup e_P$.

§5. PARTIAL COMPACTIFICATION

Firstly, as a set we have the partial compactification defined as

$$X^{BS} = X \bigsqcup_{P \subsetneq G} e_P$$

where the $\mathbb{P}$ range over all proper parabolic Subgroups of G.

To understand the picture better, let us fix a minimal parabolic $\mathbb{P}_0 \subseteq G$. Let A_0 be greatest split $\mathbb{Q}$-torus in center of the Levi component of $\mathbb{P}_0$.

Let $\Phi = \Phi(G, A_0)$ be the system of rational roots an $\Delta \subseteq \Phi$ simple roots

For any "standard" parabolic $\mathbb{P}$, let Δ_P be the restriction of the elements of Δ to A_P.

Now, if $\Delta_P = \{\alpha_1, \dots, \alpha_r\}$, we have the following isomorphism

$$\begin{aligned} A_P &\xrightarrow{\cong} (0, \infty)^r \\ a &\longmapsto (\alpha_1(a), \dots, \alpha_r(a)) \end{aligned}$$

We define a partial compactification of A_P as:

$$\boxed{\overline{A_P} := (0, \infty]^r}$$

And, we define

$$\boxed{X(P) := X \times_{A_P} \overline{A}_P = (X \times \overline{A}_P)/A_P}$$

Here, the action of A_P on X is the geodesic action.

Firstly, one should notice that

$$\begin{aligned} X &\hookrightarrow X(P) \\ x &\longmapsto [(x, (1, \dots, 1))] \end{aligned}$$

And,

$$\begin{aligned} e_P &\hookrightarrow X(P) \\ x &\longmapsto [(x, (\infty, \dots, \infty))] \end{aligned}$$

Moreover, it is important to notice that we have a bijection.

$$X(P) \longrightarrow e_P \times (0,\infty]^r$$

Intuitively, the best way to see this would be to fix representatives of of each element in e_P. And, then one sees that that this gives a unique representative for each element in $X(P)$..

Now, if $P \subseteq Q$ is an inclusion of standard parabolics, then $A_Q \subseteq A_P$, and the restricts of the geodesic action of A_P gives the geodesic action of A_Q.

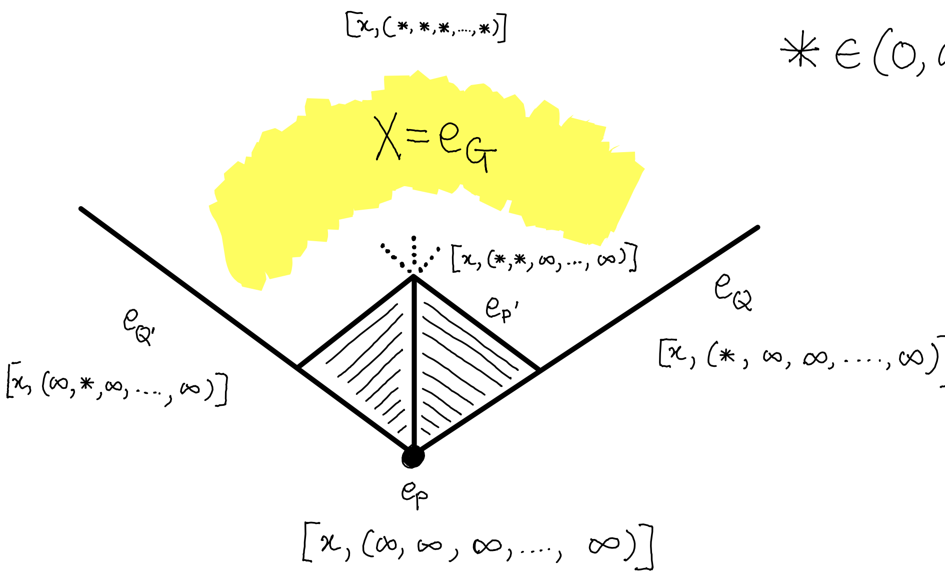
Hence $e_Q \subseteq X(P)$. Now, we want $X(P)$ to be a neighbourhood of e_P in X^{BS}

$$P \subset Q' \subsetneq P' \subseteq G$$
$$P \subsetneq Q \subsetneq P'$$

are all parabolic subgroups

X(P):

$* \in (0,\infty)$



We denote e_P as a single point merely to help us visulize

<u>Theorem</u> [Borel – Serre, §7.1]: The is a unique topology (called the Satake topology) on the union X^{BS} of X and all rational boundary components, so that the action of $G(\mathbb{Q})$ on X extends continuously to an action by homeomorphisms on X^{BS}, so that each $X(P) \subseteq X^{BS}$ is open. The parabolic subgroup P is the normalizer of the boundary component e_P.

The closure $\bar{e}_P$ of e_P in X^{BS} is the Borel-Serre partial compactification of e_P

§6. THE COMPACTIFICATION

With notation as before, let

$$\kappa : X^{BS} \longrightarrow \Gamma \backslash X^{BS} = X_\Gamma^{BS}$$

Defintion: The space X_Γ^{BS} is said to be the Borel-Serre compactification of X_Γ. And $Y_P = \kappa(e_P)$ is called a Borel-Serre stratum.

Thankfully, this doesnt destroy the local picture around e_P. Precisely, there is a neighbourhood V of e_P in X^{BS} such that:

(i) Two points in V are identified under Γ iff they are identified und $\Gamma_P = \Gamma \cap P$.

(ii) The neighborhood V is preserved by the geodesic action of $A_P^{\geq 1}$

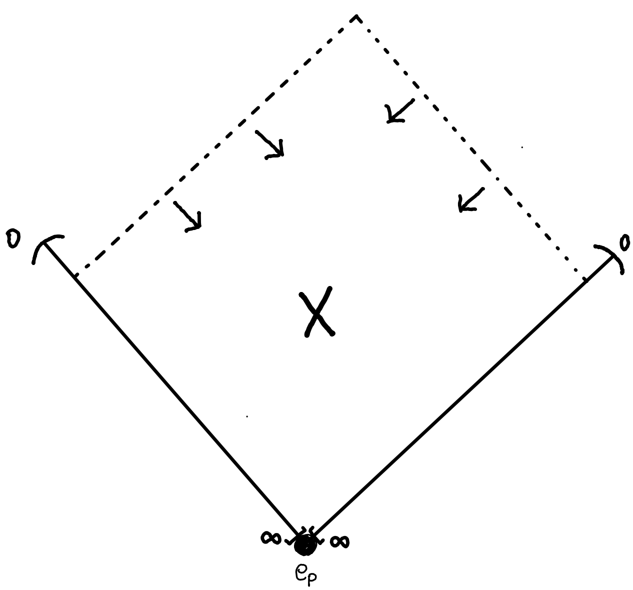
V and $\kappa(V)$ are called parabolic neighbourhood.

Here, $A_P^{\geq 1} = \{ a \in A_P \mid \alpha(a) \geq 1 \ \forall \alpha \in \Delta_P \}$.

The intersection $\kappa(V) \cap X_\Gamma$ is diffeomorphic to ${}_{\Gamma_P}\backslash X$.

For the sake of making it easier to draw, let us assume $\Delta_P = \{\alpha_1, \alpha_2\}$ has exactly two elements. Then, P is contained in exactly two distinct proper parabolic subgroups.

$$P \subseteq Q_1 \subseteq G$$
$$P \subseteq Q_2 \subseteq G$$



The action of $A_P^{<1}$ takes the points in $X(P)$ toward e_P

Finally,

Theorem (Borel-Serre): The quotient X_Γ^{BS} is compact. It is stratified with finitely many strata $Y_P = {}_{\Gamma_P}\backslash e_P$, one for each Γ-conjugacy class of rational parabolic subgroups $\mathbb{P} \leq G$. Each strata Y_P has a parabolic neighbourhood V diffeomorphic to $Y_P \times (0,\infty]^r$ (where r is the rank of A_P) whose faces $Y_P \times (0,\infty)^s$ are the intersections $Y_{P'} \cap V$ for appropriate $\mathbb{P}' \geq \mathbb{P}$.

END OF TALK 6