Langlands functoriality

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Contents

I made these notes for a seminar I gave on Langlands functoriality conjectures on the 22^{nd} of January 2024, at the University of Paderborn. I sincerely hope there are no errors in this document, but if you do find any, I would be glad if you could let me know.

Notation

For any Galois field extension E'/E, we shall denote by $\mathrm{Gal}(E'/E)$ the Galois group of the extension. Additionally, $\overline{E}$ will represent a separable algebraic closure of E, and we shall denote the absolute Galois group by $\mathrm{Gal}_E = \mathrm{Gal}(\overline{E}, E)$.

Throughout these notes K will represent a non-archimedean local field (I assume that non-archimedean fields are complete with respect to their norms), with $\mathcal{O}_K = \{x \in K : |x| \leq 1\}$ the ring of integers of K, $m_K = \{x \in K : |x| < 1\}$ the maximal ideal of $\mathcal{O}_K$ and $\varpi \in m_K$ a uniformizer of K, i.e. $m_K = (\varpi_K)$. We shall denote the residue field by $k = \mathcal{O}_k/m_k$. The characteristic of k will be denoted by p and the cardinality will be denoted by q.

On the other hand, F shall always denote a global number field, i.e. $F/\mathbb{Q}$ is a finite extension, with its ring of integers denoted by $\mathcal{O}_F$. For such a number field, we shall denote the ring of adèles by $\mathbb{A}_F = \prod'(F_\nu, \mathcal{O}_{F_\nu})$ which runs over all the places ν of F.

Over a field E, we shall denote by $\mathbb{G}_a \cong \operatorname{Spec}(E[t])$ the additive group scheme, and by $\mathrm{GL}_n \cong \operatorname{Spec}(E[\{t_{ij}\}_{1\leq i,j\leq n}, s]/(\det\{t_{ij}\}s - 1))$ the general linear group scheme (which might also be denoted by $\mathrm{GL}(V)$ for some finite dimensional group scheme).

1 Automorphic representations

I must start these notes with a apology because this first section would be rather dry with very little precision. We will be picking up from where the last talk ended, and quickly define what an automorphic representation is, without really exploring any of the details.

1.1 Admissible representations of reductive groups over local fields

First, let E be either one of $\mathbb{R}$ or $\mathbb{C}$ and G be an connected reductive group over E. We can think of $G(E)$ as real Lie group (using Weil restriction if $E = \mathbb{C}$), and $\mathfrak{g} = \operatorname{Lie}\operatorname{Res}_{E/\mathbb{R}} G$. Additionally, let $U_\infty \subseteq G(E)$ be a maximal compact subgroup of $G(E)$.

We shall define a $(\mathfrak{g}, U_\infty)$-*modules* to be a complex vector space V along with a pair of representations (which we shall abuse notation represent them both by π):

(i) $\pi : \mathfrak{g} \to \operatorname{End}(V)$ a Lie algebra representation.

(ii) $\pi : U_\infty \to \mathrm{GL}(V)$ a represenation which is locally finite and continuous.

such that:

(i) $\pi(k)\pi(X)\pi(k^{-1}) = \pi(\operatorname{ad}(k)(X))$ for all $X \in \mathfrak{g}$ and $k \in U_\infty$;

(ii) If $X \in \operatorname{Lie}(U_\infty)$ and $v \in V$, then $\pi(X)v = \lim_{t\to 0} \frac{\pi(\exp(\mathrm{tX}))v - v}{t}$.

Let V as above be a $(\mathfrak{g}, U_\infty)$-modules. Given any finite dimensional representation σ of U_∞, we can define something called an *isotropic subspace* of V as:

$$V(\sigma) = \operatorname{Image}(\operatorname{Hom}_{U_\infty}(\sigma, V) \otimes \sigma \to V).$$

We say that V is *admissible* if all of these isotropic subgroups are finite dimensional.

Now that we have defined admissible for the archimedean local field case, we define the same term in non-archimedean local field case. Suppose G is a connected reductive group over a non-archimedean local field K. A representation π of $G(K)$ on a complex vector space V is said to be *admissible* if:

(i) for all $v \in V$, $\operatorname{Stab}_{G(K)}(v)$ is open in $G(K)$;

(ii) for all open subgroup $U \subseteq G(K)$, the subspace V^U is finite dimensional.

Given an irreducible admissible representation (π, V), one can check that the center $C(G(K))$ acts by scalars on V, and we call the corresponding character of $C(G(K))$ the *central character* of π and denote it by ω_π. Associated to the representation π, there is something called *contragradient representation* denoted by $\pi^\vee$ on $V^\vee$. This is defined in such a way that it acts as a dual to π, i.e. there is a natural isomorphism $\pi \xrightarrow{\cong} \pi^{\vee\vee}$. Then, for any $f \in V^\vee$ and $v \in V$, we define normalised matrix coefficients $c_{f,v} : {}^{G(K)}/_{C(G(K))} \to \mathbb{R}_{\geq 0}$ which takes $g \mapsto \frac{|f(\pi(g)v)|}{\omega_\pi(g)}$.

Suppose μ is some Haar measure on ${}^{G(K)}/_{C(G(K))}$. We say the π is a *discrete series/square integrable representation* if

$$\int_{G(K)/C(G(K))} c_{f,v}(g)^2 d\mu < \infty.$$

And, we say that it is *tempered* if for all $\epsilon > 0$

$$\int_{G(K)/C(G(K))} c_{f,v}(g)^{2+\epsilon} d\mu < \infty.$$

1.2 Automorphic representations in the adèlic perspective

In this section we shall try and put together the two definitions given in the last section. Let G be a reductive group over a number field F with a reductive integral model $\mathcal{G}$ over $\mathcal{O}_F[1/N]$ for some positive integer N. We shall denote the ring of adèles of F by $\mathbb{A}_F$. First and foremost, we can break this ring of adèles into its finite part $\mathbb{A}_{F,f}$ and infinite part F_∞, i.e. $\mathbb{A}_F = \mathbb{A}_{F,f} \times F_\infty$.

Next, we define an *admissible* representation of $G(\mathbb{A}_F)$ on a complex vector space V to be a pair of representations (which like before we shall represent both using π):

(i) $\pi : G(\mathbb{A}_{F,f}) \times U_\infty \to \mathrm{GL}(V)$ a group homomorphism,

(ii) $\pi : \mathfrak{g} := (\mathrm{Lie}\, G(F_\infty))_{\mathbb{C}} \to \mathrm{End}(V)$ a Lie algebra representation.

such that:

(i) for all $v \in V$, $\mathrm{Stab}_{G(\mathbb{A}_{F,f})}(v)$ is open in $G(\mathbb{A}_{F,f})$;

(ii) $\pi|_{U_\infty}$ is locally finite and continuous;

(iii) If $X \in \mathrm{Lie}(U_\infty)$ and $v \in V$, then $\pi(X)v = \lim_{t\to 0} \frac{\pi(\exp(\mathrm{tX}))v - v}{t}$;

(iv) $\pi(g)\pi(X)\pi(g^{-1}) = \pi(\mathrm{ad}(g_\infty)(X))$ for all $X \in \mathfrak{g}$ and $g = (g_f, g_\infty) \in G(\mathbb{A}_{F,f}) \times U_\infty$;

(v) For all $U \subseteq G(\mathbb{A}_{F,f})$ open subgroups, and σ an irreducible representation of U_∞, $V^U(\sigma)$ is finite dimensional.

If (π, V) is an irreducible admissible representation of $G(\mathbb{A}_F)$, the we can in fact write π as a tensor product $\pi = \pi_f \otimes \pi_\infty$, where π_f is the finite part and π_∞ is infinite part. Infact, we get that

$$\pi_f = \otimes'_{\nu\nmid\infty} \pi_\nu$$

is a restricted tensor product which runs over all the finite places of F such that π_ν is unramified (this will be discussed later, but it basically means $\pi^{\mathcal{G}(\mathcal{O}_{F_\nu})} \neq 0$) for all but finitely many places. And,

$$\pi_\infty = \otimes_{\nu|\infty} \pi_\nu,$$

where each π_ν is an irreducible admissible $(\mathfrak{g}_\nu, U_{\infty,\nu})$-module.

The space of automorphic forms for G/F, which we shall denote by $A(G(F)\backslash G(\mathbb{A}_F))$, is a smooth (which is basically the same definition as admissible except for condition number (v)) representation of $G(\mathbb{A}_F)$ as follows: If $f \in A(G(F)\backslash G(\mathbb{A}_F))$, then an element $g \in G(\mathbb{A}_{F,f}) \times U_\infty$ acts via $(gf)(h) = f(hg)$ and an elements $X \in \mathfrak{g}$ acts via $(Xf)(h) = \lim_{t\to 0} \frac{f(h\exp(tX)) - f(h)}{t}$.

Definition 1.1. *An automorphic representation of $G(\mathbb{A}_F)$ is an irreducible admissible $G(\mathbb{A}_{F,f}) \times (\mathfrak{g}, U_\infty)$-module isomorphic to a subquotient of $A(G(F)\backslash G(\mathbb{A}_F))$.*

We shall denote the set of all automorphic representation of $G(\mathbb{A}_F)$ by $\mathcal{A}(G)$ or $\mathcal{A}(G(\mathbb{A}_F))$. While the nitty-gritties of automorphic representations may not be very important in the context of this talk, what will definitely prove important is the structure theorem we discussed above (which I believe is called Flath's theorem), i.e. any element $\pi \in \mathcal{A}(G)$ can be written as a restricted tensor

$$\pi = \otimes'_\nu \pi_\nu$$

where all but finitely many places the representation π_ν of $G(F_\nu)$ is unramified.

2 Langlands dual group

To define the Langlands dual group, one first needs some results about the classification of reductive groups over an algebraically closed field. Throughout this section we shall consider E to be either a local or global field, and Gal_E will be the absolute Galois group of E. Over a separable algebraic closure $\overline{E}$ of E, all reductive groups split and are classified by its root datum. While we give a short exposé here, more details can be found in [Cog04], [Bor79] and [Spr98].

There is a canonical isomorphism between isomorphism classes of connected reductive groups over $\overline{E}$, and isomorphism classes of root systems. Given such a connected reductive group G and a maximal split torus T, we define its root system to be $\psi(G) = (X^*(T), \phi, X_*(T), \phi^\vee)$, where $X^*(T)$ and $X_*(T)$ are group of characters and cocharacters of T repectively, ϕ and $\phi^\vee$ are the root system and coroot system associated to G and T respectively. The choice of Borel is equivalent to a basis $\Delta \in \phi$, and we get a correspondence between isomorphism classes of triples (G, B, T) and based root data $\psi_0(G) = (X^*(T), \Delta, X_*(T), \Delta^\vee)$. In fact, we an exact sequence

$$1 \to \mathrm{Int(G)} \to \mathrm{Aut}(G) \to \mathrm{Aut}(\psi_0(G)) \to 1.$$

Which splits under some choice of root vectors.

Now, if we consider a connected reductive group G defined over E, then there is a natural homomorphism

$$\mathrm{Gal}_E \to \mathrm{Aut}(\psi_0(\mathrm{G}))\,.$$

This gives us the first piece information to define the Langlands dual group of G. The dual based root datum of G is defined to be $\psi_0(G)^\vee = (X_*(T), \Delta^\vee, X^*(T), \Delta)$, and we define the connected dual group to be the triplet $(\hat{G}, \hat{B}, \hat{T})$ of a connected reductive Lie group, a Borel and split torus defined over $\mathbb{C}$ (this triplet is also often denoted as $({}^LG^\circ, {}^LB^\circ, {}^LT^\circ)$). Since we have the equalities:

$$\mathrm{Aut}(\psi_0(\hat{\mathrm{G}})) = \mathrm{Aut}(\psi_0(\mathrm{G})^\vee) = \mathrm{Aut}(\psi_0(\mathrm{G}))\,.$$

This gives us an action

$$\mathrm{Gal}_E \to \mathrm{Aut}(\hat{G}).$$

Definition 2.1. *Given a connected reductive group G over a local or global field E, we define the Langlands dual group or the L-group of G to be the semidirect product*

$${}^LG := \mathrm{Gal}_E \ltimes \hat{G}.$$

Remarks 2.2. *1. Since G splits over some finite Galois extension E'/E, one easily checks that the group $\mathrm{Gal}_{E'}$ acts trivially on $\hat{G}$, and so that action passes through the finite group $\mathrm{Gal}(E'/E)$. Later we see that it better to consider the elements of $\mathrm{Gal}(E'/E) \ltimes \hat{G}$ in the context of Satake parameters without it creating any major issues.*

2. *Some texts also define the L-group to be*

$$W_E \ltimes \hat{G},$$

where W_E *is Weil group (which we shall define in the next section), but this does not create any major issues because* W_E *is dense in* Gal_E *and so we can get the desired quotients discussed in the previous remark.*

3 Satake correspondence

3.1 Local non-archimedean case

One of the many benefits of the Satake isomorphism is that it gives us a way to reformulate Langlands functoriality conjectures using families of conjugacy classes called Satake parameters. First, let us understand these in the split reductive group case.

Here, we shall assume that G is a connected split reductive group over a non-archimedean local field K with maximal split torus T. Then G is the generic fiber of a group scheme $\mathcal{G}$ over $\mathcal{O}_K$ with reductive special fiber. We shall denote by $\mathcal{U} = \mathcal{G}(\mathcal{O}_K) \subseteq G(K)$ a hyperspecial maximal compact subgroup of $G(K)$.

Now we define the *Hecke ring* of G which we shall denote by $\mathcal{H}_G = \mathcal{H}(G(K), \mathcal{U})$ as the set of all locally constant, compactly supported function $f : G \to \mathbb{Z}$ which are bi-$\mathcal{U}$-invariant. The functions can hence be viewed as:

$$f : \mathcal{U} \backslash G(K)/\mathcal{U} \to \mathbb{Z}.$$

One can define a convolution product on $\mathcal{H}_G$ giving it the (as of now unknown if commutative) ring structure using the unique Haar measure on G giving $\mathcal{U}$ volume 1. More generally, so the Hecke ring is sometimes defined without the bi-$\mathcal{U}$-invariant condition and the $\mathcal{H}_G$ is viewed as subring of that. Some also define the Hecke ring as a collection of measures on G with additional properties, but we are following [Gro98] for this section and shall stick to this definition. We can also define $\mathcal{H}_T$ in a similar manner.

The Satake transform defines a ring homomorphism

$$\mathcal{S} : \mathcal{H}_G \to \mathcal{H}_T \otimes \mathbb{Z}[q^{\pm 1/2}].$$

Intuitively, the tensor product above can be viewed as functions from $T(K)$ to $\mathbb{Z}[q^{\pm 1/2}]$ with the additional properties of the Hecke ring. As mentioned above, it isn't clear if $\mathcal{H}_G$ is in fact commutative, but it is not hard to show that $\mathcal{H}_T$ is commutative and in fact isomorphic to $\mathbb{Z}[X_*(T)]$, where $X_*(T)$ are as seen previously, the co-characters of T. One of the nice consequences of the Satake transform is that $\mathcal{H}_G$ is in fact commutative.

The Weyl group $W = N_G(T)/T$ has a natural action on the Hecke ring $\mathcal{H}_T$, and as discussed in [Gro98], we get an isomorphism

$$R(\hat{G}) = \mathbb{Z}[X_*(T)]^W,$$

where $R(\hat{G})$ is the Grothendieck ring of finte dimensional complex algebraic representations of the connected dual group $\hat{G}$ of G. A consequence of this as discussed in [Car+79] and [Sat63] is:

Proposition 3.1. *The Satake transform gives a ring isomorphism*

$$S : \mathcal{H}_G \otimes \mathbb{Z}[q^{\pm 1/2}] \xrightarrow{\cong} R(\hat{G}) \otimes \mathbb{Z}[q^{\pm 1/2}].$$

In the context of Langlands functoriality, the above isomorphism is important because the characters of representation ring

$$R(\hat{G}) \otimes \mathbb{C} \to \mathbb{C}$$

are indexed by semisimple conjugacy classes s in the connected dual group $\hat{G}(\mathbb{C})$. Since we have fixed an isomorphism between $\mathcal{H}_G \otimes \mathbb{C} \cong R(\hat{G}) \otimes \mathbb{C}$, we see that character of the Hecke ring $\mathcal{H}_G$ are indexed by semisimple conjugacy classes in $\hat{G}(\mathbb{C})$.

An irreducible smooth complex representation of $G(K)$, π is said to be *unramified* if the fixed points $\pi^{\mathcal{U}} \neq 0$. An interesting fact about such an unramified representation is that $\dim_{\mathbb{C}}(\pi^{\mathcal{U}}) = 1$. More generally, one can define unramified representation when the reductive group G is quasi-split and splits over some unramified extension of K, in such a case the reductive group is said to be *unramified*. This will be important for us only in the next section, so we return back to a split reductive group for now.

Now, the Hecke ring $\mathcal{H}_G$ has a natural action on a smooth irreducible representations defined as follows. Given a $v \in \pi$ and $f \in \mathcal{H}_G$, we can find a compact open $\mathcal{U}_0 \subseteq G(K)$ such that $u.v = v$ and $f(g.u) = f(g)$ for all $u \in \mathcal{U}_0$, and $g \in G(K)$. Then we define

$$f \cdot v = \sum_{g \in G(K)/\mathcal{U}_0} \mu(U_0) f(g)(g \cdot v),$$

where μ is the Haar measure discussed above with $\mu(\mathcal{U}) = 1$.

Now, if π is unramified as discussed above, then we get a character $\mathcal{H}_G \otimes \pi^{\mathcal{U}} \to \pi^{U}$, where $\pi^{\mathcal{U}} \cong \mathbb{C}$. This gives us a semisimple conjugacy class $s(\pi)$ in $\hat{G}(\mathbb{C})$ called the *Satake parameter* of π. The association is so strong it infact gives us the following result which is proved in [Car+79]:

Proposition 3.2. *The map $\pi \mapsto s(\pi)$ gives a bijection between the set of isomorphism classes of unramified irreducible representation of $G(K)$ and the set of semisimple conjugacy classes in $\hat{G}(\mathbb{C})$.*

3.2 Global number field case

We now move to case where G is defined over a number field F. For any place ν of F, as mentioned earlier, it is possible to define an unramified representation of $G(F_\nu)$ when G is unramified over F_ν (defined in the last section). There exists some finite Galois extension E/F such that G splits over E, and we know that the above condition is satisfied for all but finitely many places of F. This means that for all but finitely many places of ν of F, we can talk about unramified representations of $G(F_\nu)$. For such a place ν, we get a bijection between unramified irreducible representations π_ν of $G(F_\nu)$ and a semisimple conjugacy class $c(\pi_\nu)$ in

$$\mathrm{Gal}(E/F) \ltimes \hat{G},$$

whose image in $\mathrm{Gal}(E/F)$ equals the Frobenius class of ν. Note that $\mathrm{Gal}(E/F) \ltimes \hat{G}$ is a reductive group over $\mathbb{C}$, and hence we have the standard results of semisimplicity of reductive group. Additionally, the Frobenius class in $\mathrm{Gal}(E/F)$ is all also well defined for all but finitely many places.

What makes this above parameterization interesting is that, given an automorphic representation π we now have a family of conjugacy classes in $\mathrm{Gal}(E/F) \ltimes \hat{G}$:

$$c(\pi) = \{c(\pi)_\nu = c(\pi_\nu)\}_{\nu \notin S}$$

defined at all but finitely many places S of the number field F. The set S does not play a major role in the theory, and so we can define these families up to equivalence where $c(\pi)$ and

$c'(\pi)$ are equivalent if for almost all places ν of F, $c(\pi)_\nu = c'(\pi)_\nu$. This is discussed in more detail with concrete examples in [Art03].

As we will see in the next section, functoriality is trying to discuss the relationship between automorphic representations defined over different reductive groups. The discussion in this chapter suggests that we could potentially rephrase that using these Satake parameters.

4 Local and global Langlands conjectures and automorphic L-functions

4.1 Local Langlands conjecture

We shall first discuss the local Langlands conjecture, and for that we denote by K a local field over $\mathbb{Q}_p$ for some prime number p with residue field k (the local Langlands conjecture can also be stated in a similar manner for archimedean fields, but we stick to non-archimedean here). Then, we have an exact sequence

$$1 \to I_K \to \mathrm{Gal}_K \xrightarrow{\phi_K} \mathrm{Gal}_k \to 1,$$

where I_K is called the *Inertia group*. A standard result from infinite dimensional Galois theory is that we have a canonical isomorphism $\mathrm{Gal}_k = \hat{\mathbb{Z}}$ of topological groups, where the arithmetic Frobenius, which we shall denote by Frob_k, is mapped to 1. We define the *Weil group* to be the group

$$W_K = \phi_K^{-1}(\mathbb{Z}),$$

whose topology is defined by two conditions. Firstly, you want I_K which possesses the subspace topology of Gal_K to be open in W_K. And secondly, you want the following exact sequence to be one of topological groups

$$1 \to I_K \to W_K \xrightarrow{\phi_K} (\mathrm{Frob}_k)^{\mathbb{Z}} \to 1,$$

where the group $(\mathrm{Frob}_k)^{\mathbb{Z}}$ is given the discrete topology.

It is clear that the map ϕ_K from above defines a norm on W_K, where for any $w \in W_K$ we define

$$||w|| = q^{-\phi_K(w)}$$

Now, the *Weil-Deligne group* of K is defined to be the semidirect product

$$W_K' = W_K \ltimes \mathbb{C}$$

with respect to the action $w \cdot z = ||w||z$. We define a *Weil-Deligne representation* on some finite-dimensional vector space V over a field E of characteristic zero to be a pair (ρ, N) where:

(i) $\rho : W_K \to \mathrm{GL}(V)$ is a representation of the Weil group whose kernel contains an open subgroup of I_K (this makes the definition independent of the topology of E).

(ii) An endomorphism N of V such that $\rho(w)N\rho(w)^{-1} = ||w||N$, for all $w \in W_K$.

A Weil-Deligne representation (ρ, N) is said to be *Frobenius semisimple* if one of the following three equivalent conditions hold:

(i) $\rho(w)$ is semisimple for all $w \in W_K$;

(ii) $\rho(\sigma)$ is semisimple for some lift σ of Frob_k in W_K;

(iii) ρ is semisimple.

Remarks 4.1. *1. Once can prove that the endomorphism N in any Weil-Deligne representation (ρ, N) is actually nilpotent. The idea of the proof is to show that if this is not true and α is a non-zero eigenvalue, then $||w||\alpha$ is also a non-zero eigenvalue for all $w \in W_K$, and this is not possible because V is finite dimensional.*

2. One often sees the Weil-Deligne group being defined as a scheme, where the Weil group is given a scheme structure by defining

$$W_K = \varprojlim_{J \subseteq I_K} \Big(\bigsqcup_{W_K/J} \operatorname{Spec} \mathbb{Q} \Big),$$

where J runs over all open normal subgroups of I_K, and $\Big(\bigsqcup_{W_K/J} \operatorname{Spec} \mathbb{Q} \Big)$ is thought of as a discrete group scheme. Then, one defines

$$W'_K = W_K \ltimes \mathbb{G}_a$$

Once can check [Tat79] for a more detailed discussion on this. But since we care about $W_K(\mathbb{C})$, we shall consider our definition instead.

3. Many texts define a Weil-Deligne representation as we have and this might seem to have little to do with the Weil-Deligne group. But, the scheme theoretic interpretation comes in handy here. If we work with the base field $\mathbb{Q}$, then one can show that all n-dimensional algebraic representation $\mathbb{G}_a \to \mathrm{GL}_n$ are parameterized by nilpotent endomorphism of $\mathbb{Q}^n$. The idea is that a nilpotent endomorphism N corresponds to the representation $\sigma_N : \mathbb{G}_a \to \mathrm{GL}_n$ which takes $\alpha \mapsto \exp(N)\alpha$. This is discussed in more generality in the blog post [Yua17]. Now an algebraic representation of the Weil-Deligne group (scheme) is a morphism $\rho : W'_K \to \mathrm{GL}_n$ classified by a representation of Weil group (scheme) and a representation of $\mathbb{G}_a$ - which corresponds to a nilpotent matrix N - such that $\rho(w)N\rho(w)^{-1} = ||w||N$. Conversely a Weil-Deligne representation (ρ, N) as above corresponds to the algebraic homomorphism $W_K \ltimes \mathbb{G}_a \to \mathrm{GL}(V)$, which takes $(w, a) \mapsto \rho(w)\exp(aN)$. Again, [Tat79] discusses this in more detail.

As before, if G is a connected reductive group defined over K and ${}^LG$ is the associated L-group, we say that a homomorphism $\phi : W'_K \to {}^LG$ is *admissible* if it satisfies the following conditions:

(i) ϕ is defined over Gal_K, i.e. the following diagram commutes:

$$\begin{tikzcd} W'_K \arrow[r, "\phi"] \arrow[d] & {}^LG \arrow[d] \\ \mathrm{Gal}_K \arrow[r, equal] & \mathrm{Gal}_K \end{tikzcd}$$

(ii) ϕ is continuous, and $\phi\big|_{\mathbb{C}}$ should be algebraic (i.e. a morphism of schemes) into $\hat{G}$ with each elements $\phi(\mathbb{C})$ unipotent, and ϕ maps semisimple elements into semi-simple elements

(iii) If $\phi(W'_K)$ is contained in a Levi subgroup of a proper parabolic subgroup P of ${}^LG$, the P is relevant.

Although it might seem like the definition of admissible representation is completely new, as discussed in [Cog03] these are precisely the Frobenius semisimple complex representations of W'_K in the case of GL_n.

Definition 4.2. *We define the set $\Phi(G)$ to be the collection of all admissible homomorphisms $\phi : W'_K \to {}^LG$ modulo inner automorphisms by elements of $\hat{G}$.*

Some properties and more references regrading these sets are given in [Cog04]. Before we discuss the local Langlands correspondence, we need for few more tools, which we shall not construct here, but instead refer the reader to [Bor79]. Firstly, given a $\phi \in \Phi(G)$, we can construct a character ω_ϕ of the center $C(G)$ of G. For any $\alpha \in H^1(W'_K, C(\hat{G}))$, we can construct a character χ_α of $G(K)$, where as before $C(\hat{G})$ is the center of $\hat{G}$. One can, in a natural way define an action of $H^1(W'_K, C(\hat{G}))$ on $\Phi(G)$ using cocycles.

Definition 4.3. *We define the set $\mathcal{A}(G) = \mathcal{A}(G(K))$ to be the collection equivalence classes of irreducible admissible complex representations of $G(K)$*

As mentioned before, the local Langlands conjecture can be stated for archimedean local fields $\mathbb{R}$ and $\mathbb{C}$, where one can again define a Weil group. In this case one defines $W'_K = W_K$. But, we shall stick to non-archimedean fields. Additionally, the reason for using the same notation as we used for automorphic representation in §1 will hopefully become clear in the section on global Langlands conjecture.

Local Langlands Conjecture. *Let K be a local non-archimedean field. Then, there is a surjective map*

$$\mathcal{A}(G) \twoheadrightarrow \Phi(G)$$

with finite fibers, where for each $\phi \in \Phi(G)$, we denote the finite fiber as $\mathcal{A}_\phi = \mathcal{A}_\phi(G)$, such that:

(i) If $\pi \in \mathcal{A}_\phi$, then the central character ω_π of π is equal to ω_ϕ.

(ii) Compatibility with twisting , i.e., if $\alpha \in H^1(W'_K, C(\hat{G}))$, and χ_α is the associated character of $G(K)$, then $\mathcal{A}_{\alpha\cdot\phi} = \{\pi\chi_\alpha | \pi \in \mathcal{A}_\phi\}$.

(iii) One element $\pi \in \mathcal{A}_\phi$ is square integrable modulo $C(G)$ if and only if all $\pi \in \mathcal{A}_\phi$ are square integrable modulo $C(G)$ if and only if $\phi(W'_K)$ does not lie in a proper Levi subgroup of ${}^LG$.

(iv) One element $\pi \in \mathcal{A}_\phi$ is tempered if and only if all $\pi \in \mathcal{A}_\phi$ are tempered if and only if $\phi(W_K)$ is bounded.

(v) If H is a reductive group and $\eta : H(K) \to G(K)$ is a K-morphism with commutative kernel and co-kernel, then there is a required compatibility between decompositions for $G(K)$ and $H(K)$. Namely, η induces a natural map ${}^L\eta : {}^LG \to {}^LH$, and if we set $\phi' = {}^L\eta \circ \phi$ for $\phi \in \Phi(G)$, then any $\pi \in \mathcal{A}_\phi(G)$ can be viewed as an $H(K)$-module, decomposes into a direct sum of finitely many irreducible admissible representations belonging to $\mathcal{A}_{\phi'}(H)$.

For each $\phi \in \Phi(G)$, the finite sets $\mathcal{A}_\phi$ are called *L-packets*. In the case of $G = \mathrm{GL}_n$ these L-packets in fact turn out to be singletons and hence we can talk about a bijection $\mathcal{A}(G) \to \Phi(G)$. Other than GL_n, this conjecture has been established for some other groups (including the archimedean case), and these groups including references for each are discussed in [Cog04].

4.2 Local L-functions

As before, we assume that G is connected reductive group over a non-archimedean local field K. The discussed parameterization of irreducible admissible representation should in theory help us define local L-functions. To try and define these, we first need to define a *representation of the L-group*, which is a continuous homomorphism

$$r : {}^L G \to \mathrm{GL}_n(\mathbb{C}),$$

whose restriction to the subgroup $\hat{G}$ is morphism of complex Lie groups. Then, for any $\phi \in \Phi(G)$ we get $r \circ \phi = (\rho, N)$, a Frobenius semisimple Weil-Deligne representation. We can attach an L-function to this Weil-Deligne representation as follows: First we define the vector space $V_N^I = (\ker N)^{\rho(I_K)}$, where V denotes the vector space $\mathbb{C}^n$. Now, if we let σ^{-1} denote a choice of geometric Frobenius, then because

$$N\rho(\sigma^{-1}) = q^{-1}\rho(\sigma^{-1})N$$

and

$$I_K \trianglelefteq W_K,$$

we have that $\rho(\sigma^{-1})$ acts on V_N^I. Then, we define

$$L(s, r \circ \phi) = \det(1 - s\rho(\sigma^{-1})|V_N^I)^{-1}.$$

In the specific case when $N = 0$, one checks that this is in fact a local Artin-Weil L-function.

Definition 4.4. *Given an admissible homomorphism $\phi \in \Phi(G)$, if $\pi \in \mathcal{A}_\phi$ is in the L-packet of ϕ and $r : {}^L G \to \mathrm{GL}_n(\mathbb{C})$ is a representation of the L-group, then we define the associated local L-function as*

$$L(s, \pi, r) := L(s, r \circ \phi).$$

This definition of L-function does not distinguish between two different representations in the same L-packet. In the version of [Cog04] that I posses, Cogdell states that: "At present these L-functions are well defined only for those π for which the parametrization is know, for example if π is unramified", but a decade on, I do not know if this is still true!

4.3 Global Langlands conjecture

We would now to discuss about the potential of a global version of local Langlands conjecture. Let F be a global number field and $\mathbb{A}_F$ as before be the rings adèles of F. For a reductive algebraic group G over F, as done in §1, we shall denote by $\mathcal{A}(G) = \mathcal{A}(G(\mathbb{A}_F))$ the set of irreducible automorphism representations $G(\mathbb{A}_F)$.

While this $\mathcal{A}(G)$ would naturally act as the global analogue for admissible representations defined in the local case, the problem arises when one tries to define $\Phi(G)$ in the global case, because there is no global analogue for the Weil-Deligne group, or is there? A more precisely statement would be that there is no known global analogue for the Weil-Deligne group.

To discuss the Global Langlands correspondence, one replaces the Weil-Deligne group with a conjectural Langlands group, often denoted as $\mathcal{L}_G$, and consider admissible homomorphisms $\mathcal{L}_G \to {}^L G$. Similar to the local case, one would like these admissible homomorphisms to classify the irreducible automorphic representations, but we do not have a precise statement yet. The article [BG10] should be a good read in this regard, but I must confess, at the time of writing this I haven't read it myself.

Although little can actually be stated, one still hopes that we have a local-global compatibility. Suppose $\pi = \otimes' \pi_\nu$ is an irreducible automorphic representation where ν runs over the places of F, then if the local Langlands conjecture is true for each $G(F_\nu)$ we should get a family $\{i_\nu \circ \phi_\nu\}_\nu$ of admissible homomorphisms

$$W'_{F_\nu} \to {}^LG_{F_\nu} \to {}^LG,$$

where $\phi_\nu : W'_{F_\nu} \to {}^LG_{F_\nu}$ is the map one gets from local Langlands and $i_\nu : {}^LG_{F_\nu} \to {}^LG$ is the natural map. A local-global compatibility would tell us that this family of maps comes from some global map

$$\phi : \mathcal{L}_G \to {}^LG.$$

4.4 Global L-functions

Sticking to the notation used in the previous section, let G be a reductive group over a number field F. As in the local case, one would like to define L-functions more generally for an irreducible automorphic representation of $G(\mathbb{A}_F)$. Although, the Langlands group in the global case is still conjectural, our desire for local-global compatibility means that have all the necessary tools to discuss global L-functions.

If $r : {}^LG \to \mathrm{GL}_n(\mathbb{C})$ as before is a representation of the L-group, then we define $r_\nu = r \circ i_\nu : {}^LG_{F_\nu} \to \mathrm{GL}_n(\mathbb{C})$ for each plave ν of F, which gives us representations for the local L-groups.

Definition 4.5. *If $\pi = \otimes'\pi_\nu$ is an irreducible automorphic representation of $G(\mathbb{A}_F)$, and $r : {}^LG \to \mathrm{GL}_n(\mathbb{C})$ is a representation of the L-group, then we define the global L-function to be*

$$L(s, \pi, r) = \prod_\nu L(s, \pi_\nu, r_\nu).$$

4.5 Partial L-functions

The one obvious problem when it comes to defining the global L-functions is that one requires the local Langlands conjecture to be true for the reductive group at each place of the number field. While one is hopeful that this will be shown to be true eventually, we already know that the local Langlands conjecture is true for unramified reductive groups. This result will help us define a partial L-function.

Suppose G is a reductive group over some number field F, and $\pi = \otimes'\pi_\nu$ is an irreducible automorphic representation, then we shall denote by $S = S(\pi)$ a finite set of places of F such that for all place $\nu \notin S$, the representation π_ν is unramified.

Definition 4.6. *If $\pi = \otimes'\pi_\nu$ is an irreducible automorphic representation of $G(\mathbb{A}_F)$, and $r : {}^LG \to \mathrm{GL}_n(\mathbb{C})$ is a representation of the L-group, then we define the partial L-function to be*

$$L^S(s, \pi, r) = \prod_{\nu \notin S} L(s, \pi_\nu, r_\nu).$$

5 Functoriality

Fundamentally, what functoriality is trying to achieve is to give a relation between the sets $\mathcal{A}(G)$ and $\mathcal{A}(H)$ (in both the local and global case) given an admissible function between the L-groups of these reductive groups.

Given reductive groups G and H over a local or global field E, we shall call these admissible functions mentioned above *L-homomorphisms* and define them as group homomorphisms ${}^LH \xrightarrow{u} {}^LG$ such that

(i) it is defined over Gal_E, i.e. the following diagram commutes

$$\begin{tikzcd} {}^L H \arrow[r, "u"] \arrow[d] & {}^L G \arrow[d] \\ \mathrm{Gal}_E \arrow[r, equal] & \mathrm{Gal}_E \end{tikzcd}$$

(ii) u is continuous,

(iii) the restriction of u to $\hat{H}$ is a complex analytic homomorphism $\hat{H} \to \hat{G}$.

These L-homomorphisms become interesting when G is quasi-split, because in such a scenario it defines a function $\Phi(H) \to \Phi(G)$ which takes $\phi \mapsto u \circ \phi$.

The Principle of Functoriality. *If E is a local (resp. global) field, H and G are connected reductive E-groups with G quasi-split, then to each L-homomorphism $u : {}^L H \to {}^L G$ there is associated a transfer or lifting of admissible (resp. automorphic) representation of $H$to admissible (resp. automorphic) representations of G.*

It should be clear from our discuss above that if we assume the local or the global Langlands conjectures, then since it is possible to get an arithmetic parametrization of $\mathcal{A}(H)$ and $\mathcal{A}(G)$, we can describe the functoriality in both cases.

5.1 Local functoriality

Suppose G and H are connected reductive groups over a non-archimedean field K. As mentioned in the beginning of this section, the idea of local functoriality is given an L-homormophsm $u : {}^L H \to {}^L G$, we would like to define a map:

$$\left\{ \begin{matrix} \text{(L-packets of) admissible representations} \\ \text{of } H(K) \end{matrix} \right\} \to \left\{ \begin{matrix} \text{(L-packets of) admissible representations} \\ \text{of } G(K) \end{matrix} \right\}$$

Of the cases in which the functoriality is well understood, one of them is the case in which both H and G are also unramified and so are the representations. Since local Langlands has already been show in these cases where the admissible representation is unramified, an irreducible unramified representation π on $H(K)$, will give what is known as an unramified parameter $\phi \in \Phi(H)$. The composition $\phi' = u \circ \phi$ gives an unramified parameter $\phi' \in \Phi(G)$, which intern defines an L-packet $\mathcal{A}_{\phi'}$. This L-packet in fact contain a unique irreducible unramified representation π' of $G(K)$. This is discussed in [Bor79].

The above example clearly show that there should be some connection between the Satake parameters we discussed earlier and the functoriality. We shall try and explore this in the next section.

5.2 Global functoriality

Again, suppose H and G are reductive groups over a global number field F. Then, given an L-homomorphism $u : {}^L H \to {}^L G$, the the global functoriality is trying to define a map:

$$\left\{ \begin{matrix} \text{(L-packets of) automorphism representations} \\ \text{of } H(\mathbb{A}_F) \end{matrix} \right\} \to \left\{ \begin{matrix} \text{(L-packets of) automorphic representations} \\ \text{of } G(\mathbb{A}_F) \end{matrix} \right\}$$

In comparison to the local case, one of the obvious difficulties in discussing a strategy to attack this problem is that we do not have a concrete formulation of the global Langlands correspondence. But, having a strong results for all finitely many local parameters will still help us go a long way.

Let us assume that G is quasi-split, and for each place ν over F, we shall represent the local L-homomorphism by $u_\nu : {}^LH_\nu \to {}^LG_\nu$. Now, for any irreducible automorphic representation $\pi = \otimes' \pi_\nu$ of $H(\mathbb{A}_F)$, we know that π_ν is unramified for all but finitely many places. And, since local Langlands is proven in these case, as discussed in the previous section we get a unique set of irreducible unramified representations π'_ν of $G(F_\nu)$ for all but finitely many places of F. As briefly mentioned previously, the local Langlands is true even the archimedean case as well, so we can actually consider the finite set of places mentioned above to be a finite set of finite places.

An irreducible automorphic representation $\Pi = \otimes' \Pi_\nu$ of $G(\mathbb{A}_F)$ is a said to be a *(weak) functorial lift* of π (with respect to u), if for all archimedean places and all but finitely many non-archimedean places $\Pi_\nu = \pi'_\nu$. We say Π is a *(strong) functorial lift* if $\Pi_\nu = \pi'_\nu$ for all place ν.

As was hinted at before, these seems to be a strong link between functoriallity and Satake parameters. Firstly, suppose E/F is some finite Galois extension such that H and G split over E. We know that the action Gal_F on $\hat{H}$ and $\hat{G}$ passes through $\mathrm{Gal}(E/F)$. There is a natural set theoretic bijection:

$$\left\{ \begin{matrix} \text{group homomorphisms } u : {}^LH \to {}^LG \\ \text{over the Galois group } \mathrm{Gal}_F \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{group homomorphisms } \overline{u} : \mathrm{Gal}(E/F) \ltimes \hat{H} \to \mathrm{Gal}(E/F) \ltimes \hat{G} \\ \text{over the Galois group } \mathrm{Gal}(E/F) \end{matrix} \right\}$$

Hence, the above L-homomorphism can be thought of as a homomorphism $\overline{u} : \mathrm{Gal}(E/F) \ltimes \hat{H} \to \mathrm{Gal}(E/F) \ltimes \hat{G}$. Now, as discussed in §3, to an automorphic representation π, we can associate a family of Satake parameters $c(\pi) = \{c(\pi)_\nu\}$. If we compose the elements of this family with u_ν, we get a new family of semisimple conjugacy classes $\{\overline{u}(c(\pi)_\nu)\}$ in $\mathrm{Gal}(E/F) \ltimes \hat{G}$. Then, one of the versions of Langlands functoriality discussed in [Lan70] is:

Conjecture. *Suppose that H and G and $\overline{u}$ are as above, and that G is quasi-split. Then, for any automorphic representation π of H, there is an automorphic representation π' of G such that for all but finitely many places:*

$$c(\pi')_\nu = \overline{u}(c(\pi)_\nu).$$

This 'basic' version of Langlands functoriality, as it called there, is discussed in a little bit more detail in [Art03].

Now, if π is an irreducible automorphic representation of $H(\mathbb{A}_F)$ and Π is the automorphic lift in $\mathcal{A}(G(\mathbb{A}_F))$. Then, for every representation $r : {}^LG \to \mathrm{GL}_n(\mathbb{C})$, we have an equality of L-functions

$$L^S(s, \pi, r \circ u) = L^S(s, \Pi, r),$$

where S is a finite (possibly empty) set of places. This essentially means that functoriallity in some sense preserves L-functions.

In the unramified case, there is a way to define L-functions using Satake parameter which we haven't discussed here (refer to [Bor79] and [Gro98]) outside a finite set of primes. From what I have understood (please do correct me if I am wrong), that definitions of L-function should match up with L^S, and the 'basic' form of the functoriality which we discussed above to should also preserve the L-functions.

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