Vector bundles on pro-finite-étale covers of elliptic curves

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Abstract

In this thesis we aim to study vector bundles on the pro-finite-étale covers of elliptic curves. Given an elliptic curve E over a complete algebraically closed extension K of $\mathbb{Q}_p$, we use the classification of vector bundles on E given by Atiyah to study vector bundles on the pro-finite-étale universal cover

$$\widetilde{\mathsf{E}} = \varprojlim_{[k]} \mathsf{E}.$$

We shall explicitly describe which vector bundles pull back to the trivial bundle and try to work out some details of the functor

$$2 - \varinjlim_{[k]} \mathrm{VB}_n(\mathsf{E}) \to \mathrm{VB}_n(\widetilde{\mathsf{E}}),$$

where $\mathrm{VB}_n(\cdot)$ is the category of rank n vector bundles on some space.

Contents

1 Introduction

The goal of this thesis is to study vector bundles on the universal pro-finite-étale cover of elliptic curves with the help of Atiyah's classification in the classical case. We shall consider these elliptic curves over a base field K, which will be a complete algebraically closed extension of $\mathbb{Q}_p$.

Atiyah's paper [Ati57] classifies all vector bundles on elliptic curves over an algebraically closed field (this classification will henceforth be referred to as the Atiyah-classification). Using Tate's version of the GAGA theorem for rigid spaces, we know that all analytic vector bundles on an elliptic curve are the same as the vector bundles on the algebraic variety.

Given an elliptic curve E over an algebraically closed non-archimedean field K, consider its pro-finite-étale universal cover $\widetilde{\mathsf{E}}$, defined in [Heu22c] as

$$\widetilde{\mathsf{E}} = \varprojlim_{\mathsf{C}\to\mathsf{E}} \mathsf{C},$$

where C runs over all connected finite-étale covers of E. This in fact can be represented by a perfectoid space. We would like to use the Atiyah-classification to study vector bundles on $\widetilde{\mathsf{E}}$.

A first step towards this goal is to understand which vector bundles on E pull back to the trivial bundle on $\widetilde{\mathsf{E}}$. When the rigid Picard variety is representable, [Heu22b] describes which line bundles pull back to the trivial bundle on $\widetilde{\mathsf{E}}$. We will give more details in §2.5. In §3.1 we will discuss the consequences of these results for elliptic curves. In the case of a Tate curve, we shall give a more explicit description of these line bundles. Suppose $\mathsf{E}_q = \mathbb{G}_m/q^{\mathbb{Z}}$ is a Tate curve corresponding to $q \in \mathsf{K}$ such that $|q| < 1$. Then, we shall show in Proposition 3.1.1 that the line bundles that pull back to the trivial bundle on the universal pro-finite-étale cover $\widetilde{\mathsf{E}_q}$ are exactly those elements of $\mathrm{Pic}^0(\mathsf{E}_q)$ which correspond to the points

$$\mathsf{T}/q^{\mathbb{Z}} \subseteq \mathsf{E}_q(\mathsf{K}),$$

where if we denote by Σ, the set of roots of unity in K,

$$\mathsf{T} = q^{\mathbb{Q}}\Big(\bigcup_{\zeta\in\Sigma} \zeta(1 + \mathfrak{m}_{\mathsf{K}}\mathcal{O}_{\mathsf{K}})\Big).$$

Once we know which line bundles pull back to the trivial line bundle, we would like to show something similar for vector bundles. Theorem 2.1.1 describes the degree zero indecomposable vector bundles of any given rank on E (its isomorphism classes are denoted by $\mathcal{E}(r, 0)$). We use this description to show in Theorem 3.1.7, that for a fixed rank we can identify those vector bundles in $\mathcal{E}(r, 0)$ which pull back to the trivial bundle on $\widetilde{\mathsf{E}}$, to the topologically torsion line bundles described in [Heu22b]. Once we have this, we should have a good description of all vector bundles on E which pull back to the trivial vector bundle on $\widetilde{E}$ (in terms of the Atiyah-classification), which we shall state and prove in Theorem 3.1.8.

We would like to expand on these results in §3.2. In this section, we will look at the functor

$$2 - \varinjlim_{[k]} \mathrm{VB}_n(E) \to \mathrm{VB}_n(\widetilde{E})$$

and try and describe a 'kernel' for this functor. In this section, we will also discuss what we understand about the essential image of this functor.

Understanding vector bundles is important in the context of the p-adic Corlette–Simpson correspondence. In the case of elliptic curves, this correspondence states the following:

Theorem. *For an elliptic curve* E *over* K*, there is an equivalence of categories:*

$$\begin{gathered}\{\textit{finite dimensional continuous } K\textit{-linear representations of } \pi_1(E, 0)\} \\ \updownarrow \\ \{\textit{pro-finite-étale Higgs bundles on } E\}\end{gathered}$$

This correspondence was constructed in more generality for abeloid varieties in [HMW22].

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2 Preliminaries

2.1 Vector bundles on elliptic curves

The paper [Ati57] describes and proves the aforementioned Atiyah classification. If E is an elliptic curve over an algebraically closed field K, then the Abel-Jacobi map $E(K) \to \mathrm{Pic}^0(E)$ is an isomorphism of groups, where $\mathrm{Pic}^0(E)$ are the isomorphism classes of the degree zero line bundles. Atiyah proves the existence of a similar bijection for indecomposable vector bundles of arbitrary degree and rank.

Firstly, we define the degree of a vector bundle to be the degree of its determinant bundle. Now, for any $r, d \in \mathbb{Z}$ with $r \geq 1$ we denote by $\mathcal{E}(r, d)$ the set of all isomorphism classes of indecomposable vector bundles of rank r and degree d on E.

Theorem 2.1.1 ([Ati57] Part II, Theorem 5).

1. *There exists a vector bundle* $F_r \in \mathcal{E}(r, 0)$*, unique up to isomorphism with* $\Gamma(E, F_r) \neq 0$ *for all* $r \in \mathbb{Z}_{\geq 1}$*. Moreover, we have an exact sequence*
$$0 \to \mathcal{O}_E \to F_{r+1} \to F_r \to 0.$$
2. *Let* $V \in \mathcal{E}(r, 0)$*, then* $V \cong \mathcal{L} \bigotimes F_r$*, where* $\mathcal{L}$ *is a line bundle of degree* 0*, unique up to isomorphism.*

This shows us that given a rank, the isomorphism classes of indecomposable degree zero vector bundles of that rank are in bijection with $\mathrm{Pic}^0(E)$. We can extend this to vector bundles of other degrees as well. But, for that we need to fix a degree one line bundle. For instance, we can define $\mathcal{L}_0 = \mathcal{O}([e])$ to be the line bundle associated to the divisor $[e]$, where $e \in E(K)$ is the identity element. Then, we have the following two results:

Theorem 2.1.2 ([Ati57] Part II, Theorem 6). *Let* $\mathcal{L}_0$ *be a fixed degree one line bundle on* E*. Then* $\mathcal{L}_0$ *determines a one-to-one correspondence*
$$\alpha_{r,d} : \mathcal{E}(h, 0) \to \mathcal{E}(r, d),$$
where $h = (r, d)$ *is the highest common factor of* r *and* d*. If we choose a representative bundle* V *from the class* $\mathcal{E}(h, 0)$*,* $\alpha_{r,d}$ *is defined uniquely by the following properties:*

1. $\alpha_{r,0} = 1$ *(the identity),*
2. $\alpha_{r,r+d}(V) \cong \alpha_{r,d}(V) \bigotimes \mathcal{L}_0$*,*
3. *if* $0 < d < r$*, we have an exact sequence:*
$$0 \to T_{(d)} \to \alpha_{r,d}(V) \to \alpha_{r-d,d}(V) \to 0,$$

where $\mathsf{T}_{(d)}$ *is the trivial bundle of rank* d*. Finally, we have that* $\det \alpha_{r,d}(V) \cong \det(V) \bigotimes (\mathcal{L}_0)^d$.

Remark 2.1.3. *For any* $r, d \in \mathbb{Z}$ *with* $r \geq 1$*, Theorem 2.1.2 tells us that any element of* $\mathcal{E}(r, dr)$ *can be written as* $V \bigotimes \mathcal{O}(d[e])$ *for some* $V \in \mathcal{E}(r, 0)$*. Theorem 2.1.1 further shows us that any element in* $\mathcal{E}(r, dr)$ *can be written as* $F_r \bigotimes \mathcal{L}$ *for some* $\mathcal{L} \in \mathrm{Pic}(E)$ *of degree d. This fact will be important later.*

Theorem 2.1.4 ([Ati57] Part II, Theorem 7)**.** *Let* E *be an elliptic curve and* $\mathcal{L}_0$ *a fixed base point on* E*. We may regard* E *as an abelian variety with* $\mathcal{L}_0$ *as the zero element. Let* $\mathcal{E}(r, d)$ *denote the set of equivalence classes of indecomposable vector bundles over* E *of rank* r *and degree* d*. Then, each* $\mathcal{E}(r, d)$ *may be identified with* $E(K)$ *in such a way that*

$$\det : \mathcal{E}(r, d) \to \mathcal{E}(1, d) \textit{ corresponds to } H : E \to E,$$

where $H(x) = hx = x + \cdots + x$ *(* h *times), where* $h = (r, d)$ *is the highest common factor of* r *and* d*.*

For any $r, d \in \mathbb{Z}$ with $r \geq 1$, we have an element $F_h \in \mathcal{E}(h, 0)$, where $h = (r, d)$ is the highest common factor as before. Then, we denote by $F_{\mathcal{L}_0}(r, d) \coloneqq \alpha_{r,d}(F_h)$ in $\mathcal{E}(r, d)$.

2.2 Rigid spaces and Tate curves

In the complex setting, Serre defined the analytification functor for complex varieties in [Ser56], which we call the GAGA functor. Tate then defined the notion of rigid spaces in the 1960s, and his work was compiled and later published in [Tat71]. These rigid spaces allowed us to define a non-archimedean analogue of the GAGA functor. Historically, the most important example of a rigid space is the Tate curve. For elliptic curves over local fields with non-integral j-invariant, Tate showed that there is a p-adic analogue of the complex uniformisation theorem (after a quadratic extension depending on the type of reduction). This uniformisation allows us to write the elliptic curve as a quotient of $\mathbb{G}_m$ by a subgroup $\{q^n : n \in \mathbb{Z}\}$ for some $q \in K$ with $|q| \leq 1$.

Throughout the rest of this document, K will represent a non-archimedean field (which we define below). In this section we wish to define and discuss properties of the aforementioned rigid spaces over these non-archimedean fields. We shall omit most proofs as they can be found in [BGR84] and/or [Bos14].

Definition 2.2.1. *A* ***non-archimedean field*** *is a field* K *along with a non-archimedean norm* $|\cdot| : K \to \mathbb{R}_{\geq 0}$ *such that*

1. $|x| = 0$ *if and only if* $x = 0$,

2. $|xy| = |x||y|$,

3. $|x + y| \leq \max\{|x|, |y|\}$.

And, K *is complete with respect to the metric defined by this norm (i.e.* $d(x, y) = |x - y|$*).*

If K is a non-archimedean field, then we denote by

$$\mathcal{O}_K := \{x \in K : |x| \leq 1\},$$

the rings of integers of K. This is also sometimes denoted by K° (the reason for which will be made clear in §2.3). This ring is in fact a local ring and we denote its maximal ideal by

$$\mathfrak{m}_K := \{x \in K : |x| < 1\}.$$

Proposition 2.2.2. *If* K *is a non-archimedean field with norm* $|\cdot|_K$ *and* L *is a finite extension, then this norm on* K *extends uniquely to a non-archimedean norm* $|\cdot|_L$ *on* L*.*

Idea of proof. For any $x \in L$, define $|x|_L := |\mathrm{N}_{L/K}(x)|_K^{1/[L:K]}$, where we denote by $\mathrm{N}_{L/K} : L \to K$, the norm function. □

The above proposition tells us that we can also uniquely extend the norm of K to a fixed algebraic closure $\overline{K}$.

Much like how affine spaces (which are of finite type for varieties) are the building blocks for schemes, affinoid spaces form the building blocks for rigid spaces. Hence, we first define the notion of an affinoid algebra.

Definition 2.2.3. *Given a non-archimedean field* K*, and a positive integer* n*, we define the* ***Tate algebra*** *to be*

$$T_n = K\langle X_1, ..., X_n\rangle := \{\Sigma a_i X_1^{\alpha_1^{(i)}} \cdots X_n^{\alpha_n^{(i)}} \in K[[X_1, ..., X_n]] : a_i \to 0 \text{ as } i \to \infty\}.$$

An ***affinoid algebra*** *is a* K*-Banach algebra* A *such that there exists a continuous epimorphism of* K*-algebras* $\alpha : T_n \to A$*.*

There is a natural K-Banach algebra norm one can define on a Tate algebra by taking

$$||\Sigma a_i X_1^{\alpha_1^{(i)}} \cdots X_n^{\alpha_n^{(i)}}|| := \sup_i |a_i|.$$

We also get a residue norm on any affinoid algebra which induces the given Banach topology on A (though clearly this depends on the choice of surjection).

One can in fact show that these affinoid algebras are noetherian. And, Banach's open mapping theorem tells us that A is isomorphic as a K-Banach algebra to the residue algebra $T_n/\ker\alpha$.

A more interesting (semi-)norm is the sup-norm (although it need not necessarily be a norm). For this, we first need the following result.

Proposition 2.2.4. *If $\mathfrak{m} \subseteq T_n$ is a maximal ideal, then the composition $K \to T_n \to {}^{T_n}/_{\mathfrak{m}}$ is a finite extension.*

Proof. Refer to [BGR84, §6.1.2 Corollary 3] or [Bos14, §2.2 Corollary 12]. $\square$

If A is an affinoid algebra and $f \in A$, we define

$$|f|_{\sup} \coloneqq \sup_{\mathfrak{m} \in \mathrm{MaxSpec}(A)} |f(\mathfrak{m})|$$

where $f(\mathfrak{m})$ is the image of f in ${}^{T_n}/_{\mathfrak{m}}$.

There is a rigid geometric version of Hilbert's Nullstellensatz which says that $\mathrm{MaxSpec}\, T_n = {}^{\mathcal{O}^n_{\overline{K}}}/_{\mathrm{Gal}(\overline{K}/K)}$. Now, given an affinoid algebra A, we denote by $\mathrm{Sp}\, A$ the set of all maximal ideals of A. Using the fact that there is a natural map $\mathrm{Sp}\, A \hookrightarrow \mathrm{Sp}\, T_n = {}^{\mathcal{O}^n_{\overline{K}}}/_{\mathrm{Gal}(\overline{K}/K)}$, we can define a canonical topology on $\mathrm{Sp}\, A$ using the quotient topology of $\mathcal{O}^n_{\overline{K}}$. We shall define morphisms $\mathrm{Sp}\, A \to \mathrm{Sp}\, A'$ to be those that correspond to continuous ring homomorphisms $A' \to A$.

Definition 2.2.5. *Let A be an affinoid algebra. We say that a subset $U \subseteq \mathrm{Sp}\, A$ is an* ***affinoid subdomain*** *if there exists an affinoid algebra A' and a morphism $\mathrm{Sp}\, A' \to \mathrm{Sp}\, A$ such that it passes through U, with the property that for any morphism $\mathrm{Sp}\, B \to \mathrm{Sp}\, A$ which passes through U, also factors through $\mathrm{Sp}\, A'$.*

$$\begin{tikzcd} & \mathrm{Sp}\, B \arrow[dl, dashed] \arrow[dr] & \\ \mathrm{Sp}\, A' \arrow[rr] & & \mathrm{Sp}\, A \end{tikzcd}$$

Proposition 2.2.6. *If $U \subseteq \mathrm{Sp}\, A$ is an affinoid subdomain with the universal morphism defined above being $i : \mathrm{Sp}\, A' \to \mathrm{Sp}\, A$, then:*

1. *i defines a bijection from $\mathrm{Sp}\, A'$ to U.*

2. *U is an open subset of $\mathrm{Sp}\, A$.*

Proof. Refer to [BGR84, §7.2.2 Proposition 1 and §7.2.5 Theoem 3] or [Bos14, §3.3 Lemma 10 and Proposition 19]. $\square$

Now, the canonical topology of $\mathrm{Sp}\, A$, while a natural choice, happens to be too fine for practical purposes. The fact that the open unit disk turns out to not be connected, makes defining analytic functions very difficult. On the other hand, the Zariski topology on the maximal spectrum happens to be too coarse. Hence, we define a Grothendieck topology on $\mathrm{Sp}\, A$ that fits in the middle.

Definition 2.2.7. *Let* $\mathsf{X} = \operatorname{Sp} \mathsf{A}$*, and* $\mathsf{U} \subseteq \mathsf{X}$ *be an open subset.*

1. *We say that* U *is* ***admissible open*** *if there exists a covering* $\mathsf{U} = \bigcup_{i \in I} \mathsf{U}_i$ *by affinoid subdomains, such that for any morphism* $\phi : \operatorname{Sp} \mathsf{B} \to \operatorname{Sp} \mathsf{A}$ *which passes through* U*, the covering* $\{\phi^{-1}(\mathsf{U}_i)\}$ *has a finite refinement by affinoid open subsets.*

2. *We say that a covering* $\mathsf{U} = \bigcup \mathsf{U}_i$ *is an* ***admissible cover*** *if all* U_i*s and* U *are admissible opens and given any morphism* $\phi : \operatorname{Sp} \mathsf{B} \to \operatorname{Sp} \mathsf{A}$ *which passes through* U*, the covering* $\{\phi^{-1}(\mathsf{U}_i)\}$ *has a finite refinement by affinoid open subsets.*

The **strong Grothendieck topology** on the category of admissible opens of $\operatorname{Sp} \mathsf{A}$ is defined by considering the coverings to be all the admissible coverings as defined above. Then, using Tate's acyclicity theorem ([Bos14, Theorem 4.3.1] gives a proof of the same) one can define a structure sheaf $\mathcal{O}_\mathsf{X}$ on $\mathsf{X} = \operatorname{Sp} \mathsf{A}$ such that for an affinoid subdomain, say $\mathsf{U} = \operatorname{Sp} \mathsf{B} \subseteq \mathsf{X}$, $\mathcal{O}_\mathsf{X}(\mathsf{U}) = \mathsf{B}$. Such an $\mathsf{X} = \operatorname{Sp} \mathsf{A}$ along with its structure sheaf is called an **affinoid rigid K-space**.

Note: Throughout the rest of this section, the term 'Grothendieck topology on a set X' is always defined on a category whose objects are some (not necessarily all) subsets of X and morphisms are inclusions. Objects of this category are called admissible opens and coverings are called admissible coverings.

Definition 2.2.8. *Consider a Grothendieck topology defined on some set* X*. We define three properties as follows:*

($\mathbf{G_0}$) $\emptyset$ *and* X *are admissible opens.*

($\mathbf{G_1}$) *Let* $(\mathsf{U}_i)_{i \in I}$ *be an admissible covering of an admissible open subset* $\mathsf{U} \subseteq \mathsf{X}$*. Furthermore, let* $\mathsf{V} \subseteq \mathsf{U}$ *be a subset such that* $\mathsf{V} \cap \mathsf{U}_i$ *is admissible open for all* $i \in I$*. Then,* V *is admissible open.*

($\mathbf{G_2}$) *Let* $(\mathsf{U}_i)_{i \in I}$ *be a covering of an admissible open subset* $\mathsf{U} \subseteq \mathsf{X}$ *by admissible open subsets* $\mathsf{U}_i \subseteq \mathsf{X}$ *such that* $(\mathsf{U}_i)_{i \in I}$ *admits an admissible covering of* U *by refinement. Then,* $(\mathsf{U}_i)_{i \in I}$ *itself is an admissible covering of* U*.*

Proposition 2.2.9. *Let* $\mathsf{X} = \operatorname{Sp} \mathsf{A}$ *be an affinoid* K*-space. Then, the strong Grothendieck topology on* X *satisfies* G_0*,* G_1 *and* G_2*.*

Proof. Refer to [Bos14, §5.1 Proposition 5]. □

Definition 2.2.10. *A* ***locally G-ringed*** $\mathbf{K}$***-space*** *is a pair* $(\mathsf{X}, \mathcal{O}_\mathsf{X})$ *where* X *is a set with a Grothendieck topology defined on it, and* $\mathcal{O}_\mathsf{X}$ *is a sheaf of* K*-algebras on* X*, such that for any* $x \in \mathsf{X}$*, the stalk* $\mathcal{O}_{\mathsf{X},x}$ *is a local ring.*

A morphism $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ of two locally G-ringed K-spaces is a pair (ϕ, ϕ^*), where $\phi : X \to Y$ is a continuous morphism with respect to the Grothendieck topologies on them, and ϕ^* is a system of K-homomorphisms $\phi^*_V : \mathcal{O}_Y(V) \to \mathcal{O}_X(\phi^{-1}(V))$ for every admissible open $V \subseteq Y$ which is compatible with respect to inclusions of admissible opens $V \hookrightarrow V' \subseteq Y$ such that for any $x \in X$, the induced morphism $\phi^*_x : \mathcal{O}_{Y,\phi(x)} \to \mathcal{O}_{X,x}$ is local.

Definition 2.2.11. *A **rigid analytic K-space** (sometimes referred to as a **rigid space** if the rest is clear from context) is a locally G-ringed K-space $(X, \mathcal{O}_X)$ such that:*

1. *The Grothendieck topology on X satisfies (G_0), (G_1) and (G_2);*
2. *X admits and admissible covering $(X_i)_{i \in I}$ such that $(X_i, \mathcal{O}_X|_{X_i})$ is an affinoid K-space for all i.*

A morphism of rigid analytic space is a morphism of locally G-ringed K-spaces.

There is a faithful functor from locally finite type K-schemes to rigid K-spaces and this functor satisfies some really nice properties. Similar to the complex case, this functor is also called the GAGA functor ([Bos14, §5.4] has an excellent exposition on the same). First, we fix some notations. Given a locally finite type scheme X, we denote by X^{rig} its corresponding rigid space. We can also apply this functor on coherent sheaves of these locally finite type schemes and we shall use the same notation there.

Theorem 2.2.12. *Let X be a proper K-scheme and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module. Then, the canonical maps*

$$H^i(X, \mathcal{F}) \to H^i(X^{\mathrm{rig}}, \mathcal{F}^{\mathrm{rig}})$$

are isomorphisms for all $i \geq 0$.

Proof. Refer to [Bos14, §6.3 Theorem 11]. □

Theorem 2.2.13. *Let X be a proper K-scheme and $\mathcal{F}'$ a coherent $\mathcal{O}_{X^{\mathrm{rig}}}$-module. Then, there is a coherent $\mathcal{O}_X$-module $\mathcal{F}$ satisfying $\mathcal{F}^{\mathrm{rig}} \cong \mathcal{F}'$. Furthermore, $\mathcal{F}$ is unique up to canonical isomorphism.*

Proof. Refer to [Bos14, §6.3 Theorem 13]. □

It is clear to see from the two theorems above that there is an equivalence of categories between that of vector bundles on a proper scheme X over K, and the category of vector bundles on X^{rig}.

One can define formal models of rigid spaces over K as certain formal schemes over $\mathcal{O}_K$. Then, we can define a generic-fibre functor from formal schemes to rigid spaces (under some paracompactnes assumptions). It is possible that different formal schemes give the same rigid space when this functor is applied, but they will be unique up to admissible formal blowups. Although Grothendieck believed that this was possible, it was Raynaud in [Ray74] who explained how it worked. An exposition of the same can be found in [Bos14, §8].

As mentioned in the start of this section, historically, the most important example of a rigid space is a Tate curve. First, if you consider an elliptic curve E over $\mathbb{C}$, then, using the complex uniformisation theorem we have that

$$E(\mathbb{C}) \cong \mathbb{C}/\Lambda \ ,$$

where Λ is a rank 2 $\mathbb{Z}$-lattice in $\mathbb{C}$. In fact, one can pick Λ in such a way as to make this an isomorphism of topological groups. Tate aimed to do something similar for local fields. To avoid having to worry about the case multiplicative reduction, we shall instead focus on an algebraically closed non-archimedean field K. Firstly, let $\mathbb{G}_{m,K}$ denote the rigid analytic space corresponding to the group variety Spec $K[t, t^{-1}]$. Then, for any non-zero $q \in \mathfrak{m}_K$, we consider the subgroup $q^{\mathbb{Z}} \subseteq \mathbb{G}_{m,K}$. A **Tate curve** is a curve defined by taking the quotient:

$$E_q = \mathbb{G}_{m,K}/q^{\mathbb{Z}}$$

in the category of K-groups. This is in fact an elliptic curve (as a rigid space) with j-invariant:

$$j(E_q) = \frac{1}{q} + 744 + 196884q + \cdots$$

It is clear that $j(E_q) \notin \mathcal{O}_K$ and hence that E_q has bad reduction. One can also show conversely that an elliptic curve with bad reduction is a Tate curve (depending on the type of bad reduction, this result holds after an extension if the field is local). We once again refer to [Bos14, §9.2] and also to [Sil09, Appendix C §14] for a more detailed exposition on the topic.

2.3 Adic and perfectoid spaces

Adic spaces are structures developed by Huber in [Hub93] and [Hub94] which helps generalise the notion of Rigid spaces and Berkovich spaces and also gives us a category in which we can define perfectoid spaces. Perfectoid spaces were introduced by Scholze in [Sch12] and have since been a useful tool in the field of p-adic geometry. The classical notion of universal covers is often replaced by certain perfectoid covers in the p-adic setting. We shall now recall some basic facts about adic and perfectoid space. We omit the proofs and refer to [Hub94] and also [Sch12] for details.

Definition 2.3.1. *A topological ring A is said to be* ***Huber*** *(or* ***f-adic****) if A contains an open subring $A_\circ$ (called a* ***ring of definition****) such that the subspace topology on $A_\circ$ is I-adic, where I is a finitely generated ideal of $A_\circ$.*

Consistent with the notation we used before, if A is a topological ring, we denote by A° the power bounded elements of A.

Definition 2.3.2. *Given a Huber ring A, we say that a subring $A^+ \subseteq A^\circ$ is a **ring of integral elements** if it is open in A and integrally closed.*

We first need to generalise the notion of affinoid algebras defined earlier.

Definition 2.3.3.

1. *A **Tate $\mathbf{K}$-algebra** is a topological K-algebra R, such that R is a Huber ring, and there exists a subring $R_\circ \subseteq R$ such that $aR_\circ$, $a \in K^\times$, forms a basis of open neighbourhoods of $R_\circ$.*

2. *An **affinoid $\mathbf{K}$-algebra** is a pair (R, R^+) consisting of a Tate K-algebra R and an open integrally closed subring $R^+ \subseteq R^\circ$.*

3. *An affinoid K-algebra (R, R^+) is said to be **topologically finite type** over (K, K°) if R is a quotient of $K\langle T_1, ..., T_n\rangle$ for some $n \in \mathbb{Z}_{\geq 1}$ and $R^+ = R^\circ$.*

Now that we have the required tools, we can define the adic spectrum of an affinoid K-algebra (A, A^+). We define

$$\mathrm{Spa}(A, A^+) \coloneqq \{x \in \mathrm{Cont}(A) : |f(x)| \leq 1 \text{ for all } x \in A^+\},$$

where $\mathrm{Cont}(A)$ is the set of all equivalence classes of continuous valuations on A, and $|f(x)|$ for any $f \in A$ denotes the image of f in the ordered group corresponding to the valuation x. Given $\mathrm{Spa}(A, A^+)$ as above, we define **rational subsets** as sets of the form

$$U = \mathrm{Spa}(A, A^+)(\tfrac{f_1,...,f_n}{f_0}) \coloneqq \{x \in \mathrm{Spa}(A, A^+) : |f_i(x)| \leq |f_0(x)| \text{ for all } i \geq 1\},$$

where $f_1, ..., f_n \in A$ generates the unit ideal and $f_0 \in A$. In fact, these rational subsets are themselves the adic spectrum of an affinoid K-algebra described in the following manner: First, consider the subalgebra $A_U = A\left[\frac{f_1}{f_0}, ..., \frac{f_n}{f_0}\right] \subseteq A[f_0^{-1}]$, and equip it with the topology making $aA_\circ\left[\frac{f_1}{f_0}, ..., \frac{f_n}{f_0}\right]$ a basis of open neighbourhoods of zero for all $a \in K^\times$. Let $A_U^+ \subseteq A_U$ be the integral closure of $A^+\left[\frac{f_1}{f_0}, ..., \frac{f_n}{f_0}\right]$ in $A\left[\frac{f_1}{f_0}, ..., \frac{f_n}{f_0}\right]$. Finally, the Huber pair we are looking for is their respective completions $(\widehat{A_U}, \widehat{A_U^+})$. These rational subsets form the basis of the topology on $\mathrm{Spa}(A, A^+)$.

Given the adic spectrum $X = \mathrm{Spa}(A, A^+)$ of an affinoid K-algebra, we can also define presheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$ such that for each rational open subset $U \subseteq X$, we have $U \cong \mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$. These need not be sheaves in general. But, they will be sheaves in the cases that we care about. In fact, [Hub94, Propositon 1.6] shows us that $\mathcal{O}_X^+$ is a sheaf whenever $\mathcal{O}_X$ is a sheaf.

Theorem 2.3.4 ([Hub94] Theorem 2.2)**.** *If* (A, A^+) *is a topologically finite type affinoid* K*-algebra, and* $X = \mathrm{Spa}(A, A^+)$*, then* $\mathcal{O}_X$ *and* $\mathcal{O}_X^+$ *are both sheaves.*

Consider the category $(\mathcal{V})$ of triples $(X, \mathcal{O}_X, \{ |\cdot(x)| : x \in X \})$ consisting of a locally ringed space $(X, \mathcal{O}_X)$ where $\mathcal{O}_X$ is a sheaf of complete topological K-algebras and continuous valuations $f \mapsto |f(x)|$ on $\mathcal{O}_{X,x}$ for every $x \in X$. Morphisms in this category are given by morphisms of locally ringed topological spaces which are continuous K-algebra morphisms on $\mathcal{O}_X$ and are compatible with the valuations.

Let (A, A^+) be an affinoid K-algebra, and $X = \mathrm{Spa}(A, A^+)$. If $\mathcal{O}_X$ is in fact a sheaf, we get such a triple $(X, \mathcal{O}_X, \{ |\cdot(x)| : x \in X \})$. Any object in the category $(\mathcal{V})$ that is isomorphic to such an object is called an **affinoid adic space**.

Definition 2.3.5. *An **adic space** over* K *is an object* $(X, \mathcal{O}_X, \{ |\cdot(x)| : x \in X \})$ *in the category* $(\mathcal{V})$ *that is locally an affinoid adic space.*

Hence, we get a functor from rigid spaces to adic spaces and henceforth we may abuse terminology and call a space rigid when in reality we mean the corresponding adic space. Rigid spaces are examples of locally noetherian adic spaces, as defined below.

Definition 2.3.6. *An adic space is said to be **locally noetherian** if it locally of the form* $\mathrm{Spa}(A, A^+)$*, where either* A *is strongly noetherian (i.e. it is a noetherian Tate rings such that for any* $n \in \mathbb{Z}_{\geq 1}$*,* $A\langle T_1, \dots, T_n\rangle$ *is also noetherian) or* A *admits a noetherian ring of definition.*

Now, we can define the notion of a perfectoid field which is crucial to the definition of perfectoid spaces (in the following two definitions, Φ denotes the Frobenius).

Definition 2.3.7. *A non-archimedean field* K *of residue characteristic* p *is said to be a **perfectoid field** if:*

1. $|K^*|$ *is non-discrete in* $\mathbb{R}$*;*
2. $\Phi : K^\circ/p \to K^\circ/p$ *where* $x \mapsto x^p$ *is surjective.*

From now on, K will represent a perfectoid field unless otherwise mentioned.

Definition 2.3.8. *A* K*-algebra* R *is a said to be a **perfectoid algebra** if:*

1. R *is a Banach algebra;*
2. $R^\circ \subseteq R$ *is open and bounded;*

3. $\Phi : {}^{R^\circ}/_{\varpi} \to {}^{R^\circ}/_{\varpi}$ *is surjective, where* $\varpi \in K^\circ$ *is a pseudo-uniformizer such that* $|\varpi| \geq |p|$.

Morphisms between perfectoid K*-algebras are continuous morphisms of* K*-algebras.*

Given a perfectoid field K, we define the **tilt** of K to be

$$K^\flat := \varprojlim_{x \mapsto x^p} K,$$

which is a perfectoid field in characteristic p. And, in a similar manner, we can also define a tilt $R^\flat$ of a perfectoid K-algebra R which ends up becoming a perfectoid $K^\flat$-algebra. In fact, we get what is known as the tilting equivalence as a consequence.

Theorem 2.3.9 ([Sch12] Theorem 1.7)**.** *There is a natural equivalence of categories, called the tilting equivalence, between the category of perfectoid* K*-algebras and the category of perfectoid* $K^\flat$*-algebras. Here, a perfectoid* K*-algebra* R *is sent to the perfectoid* $K^\flat$*-algebra*

$$R^\flat = \varprojlim_{x \mapsto x^p} R.$$

To define a perfectoid space, we first define a **perfectoid affinoid algebra** to be an affinoid algebra (R, R^+) such that R is a perfectoid K-algebra.

Theorem 2.3.10 ([Sch12] Theorem 6.3)**.** *Let* (R, R^+) *be a perfectoid affinoid* K*-algebra, and let* $X = \mathrm{Spa}(R, R^+)$ *with associated presheaves* $\mathcal{O}_X$ *and* $\mathcal{O}_X^+$*. Also, let* $(R^\flat, R^{\flat+})$ *be its tilt (here,* $R^{\flat+} = \varprojlim_{x \mapsto x^p} R^+$*), and let* $X^\flat = \mathrm{Spa}(R^\flat, R^{\flat+})$*.*

1. *We have a homeomorphism* $X \to X^\flat$ *that identifies rational subsets.*
2. *For any rational subset* $U \subseteq X$ *with tilt* $U^\flat \subseteq X^\flat$*, the complete affinoid* K*-algebra* $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ *is perfectoid, with tilt* $(\mathcal{O}_{X^\flat}(U^\flat), \mathcal{O}_{X^\flat}^+(U^\flat))$*.*
3. *The presheaves* $\mathcal{O}_X$*,* $\mathcal{O}_{X^\flat}$ *are sheaves.*
4. *The cohomology groups* $\mathrm{H}^i(X, \mathcal{O}_X^+)$ *are* $\mathfrak{m}$*-torsion for* $i > 0$*.*

Now, we can define a perfectoid space.

Definition 2.3.11. *An* ***affinoid perfectoid space*** *is an adic space isomorphic to* $\mathrm{Spa}(R, R^+)$ *for some affinoid perfectoid* K*-algebra* (R, R^+)*. A* ***perfectoid space*** *is an adic space over* K *that is locally isomorphic to an affinoid perfectoid space. Morphisms between perfectoid spaces are morphisms of adic spaces.*

We define the tilt of a perfectoid space X as a perfectoid space $\mathbf{X}^\flat$ over $\mathsf{K}^\flat$ such that there is a natural isomorphism

$$\mathrm{Hom}(\mathrm{Spa}(\mathsf{R}^\flat, \mathsf{R}^{\flat+}), \mathsf{X}^\flat) = \mathrm{Hom}(\mathrm{Spa}(\mathsf{R}, \mathsf{R}^+), \mathsf{X})$$

for all perfectoid affinoid K-algebras $(\mathsf{R}, \mathsf{R}^+)$ with tilt $(\mathsf{R}^\flat, \mathsf{R}^{\flat+})$.

Theorem 2.3.12 ([Sch12] Theorem 6.17)**.** *Any perfectoid space X over K admits a tilt $\mathsf{X}^\flat$ over $\mathsf{K}^\flat$, unique up to unique isomorphism such that:*

1. *This induces an equivalence of categories between perfectoid spaces over K and perfectoid spaces over $\mathsf{K}^\flat$.*
2. *The underlying topological spaces of X and $\mathsf{X}^\flat$ are naturally identified.*
3. *A perfectoid space is affinoid perfectoid if and only if its tilt is affinoid perfectoid.*
4. *For any affinoid perfectoid subspace $\mathsf{U} \subseteq \mathsf{X}$, the pair $(\mathcal{O}_\mathsf{X}(\mathsf{U}), \mathcal{O}_\mathsf{X}^+(\mathsf{U}))$ is a perfectoid affinoid K-algebra with tilt $(\mathcal{O}_{\mathsf{X}^\flat}(\mathsf{U}^\flat), \mathcal{O}_{\mathsf{X}^\flat}^+(\mathsf{U}^\flat))$.*

Definition 2.3.13.

1. *A morphism of adic spaces $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ is said to be **finite étale** if there is a cover of Y by open affinoids $\mathsf{V} \subseteq \mathsf{Y}$ such that the pre-image $\mathsf{U} = \mathsf{f}^{-1}(\mathsf{V})$ is affinoid and the associated morphism of affinoid K-algebras $(\mathcal{O}_\mathsf{Y}(\mathsf{V}), \mathcal{O}_\mathsf{Y}^+(\mathsf{V})) \to (\mathcal{O}_\mathsf{X}(\mathsf{U}), \mathcal{O}_\mathsf{X}^+(\mathsf{U}))$ is finite étale.*
2. *A morphism $\mathsf{f} : \mathsf{X} \to \mathsf{Y}$ of adic spaces over K is called **étale** if for any point $\mathsf{x} \in \mathsf{X}$, there are open neighbourhoods U and V of x and $\mathsf{f}(\mathsf{x})$ respectively, and a commutative diagram*

$$\begin{tikzcd} \mathsf{U} \arrow[r, "\mathsf{j}"] \arrow[dr, "\mathsf{f}|_\mathsf{U}"'] & \mathsf{W} \arrow[d, "\mathsf{p}"] \\ & \mathsf{V} \end{tikzcd}$$

where j is an open embedding and p is finite étale.

This allows us to define the finite-étale and étale sites of rigid spaces and perfectoid spaces in a natural way. We shall denote them by $\mathsf{X}_{\text{fét}}$ and $\mathsf{X}_{\text{ét}}$ respectively.

2.4 Pro-finite-étale universal cover

For the purpose of this section, we consider a connected locally noetherian adic space X over a complete algebraically closed extension K of $\mathbb{Q}_\mathsf{p}$ (we do not really need such a restrictive assumption on the field, but it makes things easier in our case). We shall discuss the pro-finite-étale site of X as defined in [Sch13]. First, for a category $\mathcal{C}$, we define pro $-\mathcal{C}$, the category of pro-objects.

Definition 2.4.1. *Given a category $\mathcal{C}$, we define the category* **pro** $-\mathcal{C}$ *whose objects are functors*

$$\mathsf{F} : \mathrm{I} \to \mathcal{C}$$

from small cofiltered index categories, and whose morphisms are given by

$$\mathrm{Hom}(\mathsf{F}, \mathsf{G}) = \varprojlim_{\mathrm{J}} \varinjlim_{\mathrm{I}} \mathrm{Hom}(\mathsf{F}(\mathrm{i}), \mathsf{G}(\mathrm{j}))$$

for $\mathsf{F} : \mathrm{I} \to \mathcal{C}$ *and* $\mathsf{G} : \mathrm{J} \to \mathcal{C}$ *objects in* pro $-\mathcal{C}$.

For any object, say $\mathsf{F} : \mathrm{I} \to \mathcal{C}$ in the category pro $-\mathcal{C}$, and for each object $\mathrm{i} \in \mathrm{I}$, we often denote $\mathsf{F}(\mathrm{i})$ by F_{i}, and write $\mathsf{F} = \varprojlim_{\mathrm{i}} \mathsf{F}_{\mathrm{i}}$.

Before we define the pro-finite-étale site, we need to fix some notations. If we consider the category pro $-\mathsf{X}_{\text{fét}}$, then for any object $\mathsf{U} = \varprojlim_{\mathrm{i}} \mathsf{U}_{\mathrm{i}}$ in this category, we define the associated topological space as $|\mathsf{U}| \coloneqq \varprojlim_{\mathrm{i}} |\mathsf{U}_{\mathrm{i}}|$.

Definition 2.4.2. *For* X *a connected locally noetherian adic space over* K*, we define the pro-finite-étale site* $\mathbf{X}_{\text{profét}}$ *whose underlying category is the category* pro $-\mathsf{X}_{\text{fét}}$*, and a covering is given by a family of open morphisms* $\{\mathsf{f}_{\mathrm{i}} : \mathsf{U}_{\mathrm{i}} \to \mathsf{U}\}$ *such that* $|\mathsf{U}| = \bigcup_{\mathrm{i}} \mathsf{f}_{\mathrm{i}}(|\mathsf{U}_{\mathrm{i}}|)$.

If we fix a geometric base point x of X, we can talk about the étale fundamental group $\pi_1(\mathsf{X}, x)$, which is a pro-finite group. We would like to relate the pro-finite-étale site of X with the site of pro-finite sets over $\pi_1(\mathsf{X}, x)$, which we define below.

Definition 2.4.3. *For a pro-finite group* G*, let* **G − pfsets** *be the site whose underlying category is the category of pro-finite sets* S *with a continuous* G*-action, and whose covers are families of open continuous* G*-equivariant morphisms* $\{\mathsf{f}_{\mathrm{i}} : \mathsf{S}_{\mathrm{i}} \to \mathsf{S}\}$ *such that* $\mathsf{S} = \bigcup_{\mathrm{i}} \mathsf{f}_{\mathrm{i}}(\mathsf{S}_{\mathrm{i}})$.

Now, we can state the theorem that we eluded to above.

Theorem 2.4.4 ([Sch13], Proposition 3.5)**.** *Let* X *be as above and* $x \in \mathsf{X}(\mathsf{K})$ *a geometric point. Then, there is a canonical equivalence of sites*

$$\mathsf{X}_{\text{profét}} \cong \pi_1(\mathsf{X}, x) - \text{pfsets}\,.$$

One can think of $\pi_1(\mathsf{X}, x)$ with translation as a universal object in the category in $\pi_1(\mathsf{X}, x) - $ pfsets (such a universality implicitly depends on a choice of a point for any object in $\pi_1(\mathsf{X}, x) - $ pfsets). The paper [Heu22c] defines a similar universal object in the category $\mathsf{X}_{\text{profét}}$, called the universal pro-finite-étale cover.

Definition 2.4.5. *If* X *is any proper connected rigid space over* K *and* $x \in \mathsf{X}(\mathsf{K})$*, the* ***pro-finite-étale universal cover*** $\widetilde{\mathsf{X}} \to \mathsf{X}$ *is defined as the diamond*

$$\widetilde{\mathsf{X}} = \varprojlim_{\mathsf{X}' \to \mathsf{X}} \mathsf{X}'$$

which is indexed over all connected finite étale covers $(X', x') \to (X, x)$*, where* $x' \in X'(K)$ *is a lift of* x.

Diamonds are perfectoid analogues of algebraic spaces defined in [Sch17], which have not been defined here because in the case of curves, [Han15] shows that this universal cover is in fact represented by a perfectoid space. In fact, [Bla+18] shows the following stronger result in our case of elliptic curves:

Theorem 2.4.6 ([Bla+18], Corollary 5.8)**.** *Let* A *be an abelian variety over* K*, then the perfectoid tilde-limit* [1]

$$\widetilde{A} = \varprojlim_{[k]} A,$$

which ranges over all $k \in \mathbb{Z}_{\geq 1}$*, is the pro-finite-étale universal cover.*

2.5 Pro-finite-étale line bundles

Given an elliptic curve E over a complete algebraically closed extension K of $\mathbb{Q}_p$, we are interested in studying vector bundles on its pro-finite-étale universal cover. But, first we need some results about line bundles.

Definition 2.5.1. *Let* X *be a proper connected locally noetherian adic space. We say that an analytic vector bundle* V *on* X *is* ***pro-finite-étale*** *if it is trivialized by a pro-finite-étale cover. Equivalently, it is trivialised by the pro-finite-étale universal cover* $\widetilde{X}$.

The paper [Heu22b] explicitly describes the pro-finite-étale line bundles for a proper rigid space. To understand this, we need to first define the v-topology as defined in [Sch17] and certain other structures.

Definition 2.5.2. *Let* Perf *be the category of perfectoid spaces. Then, the* ***v-site*** *is a Grothendieck topology on* Perf *where a collections of morphisms* $\{X_i \xrightarrow{f_i} X\}_{i \in I}$ *is a covering if for each quasicompact open* $U \subseteq X$*, there exists a finite subset* $J \subseteq I$ *and quasicompact opens* $U_i \subseteq X_i$ *for each* $i \in J$*, such that* $U = \bigcup_{i \in J} f_i(U_i)$.

We call a point x in a topological abelian group G **topologically torsion** if $x^{n!} \to 0$ and $n \to \infty$, and we denote by $G^{tt} \subseteq G$ the subgroup of all topologically torsion elements. We would like to define a "sheaf analogue" of this. Given an abelian v-sheaf F, we first consider the internal Hom sheaf $\underline{\mathrm{Hom}}(\widehat{\mathbb{Z}}, F)$, where $\widehat{\mathbb{Z}} = \varprojlim_{n \in \mathbb{Z}_{\geq 1}} \underline{\mathbb{Z}/n\mathbb{Z}}$ is the pro-finite sheaf.

[1] normal limits need not always exist and hence we often need to use tilde-limits as defined in [Hub13]

Definition 2.5.3. *Let* F *be an abelian v-sheaf. There is a natural evaluation map at* $1 \in \widehat{\mathbb{Z}}(\mathsf{K})$*:*

$$e_{\mathsf{F}} : \underline{\mathrm{Hom}}(\widehat{\mathbb{Z}}, \mathsf{F}) \to \mathsf{F}.$$

We define the ***topologically torsion subsheaf*** *of* F *to be the abelian subsheaf:*

$$\mathsf{F}^{tt} \coloneqq \mathrm{image}(e_{\mathsf{F}}) \subseteq \mathsf{F}.$$

Then, [Heu22b] shows us that for both the v-topology and the étale topology

$$\mathrm{R}^1 \pi_*(\mathbb{G}_m^{tt}) \cong (\mathrm{R}^1 \pi_* \mathbb{G}_m)^{tt},$$

where $\pi : X \to \mathrm{Spa}(K, K^+)$ is the structure map. This is a very strong result for our case of elliptic curves, because it gives us the following corollary:

Theorem 2.5.4 ([Heu22b] Corollary 3.7)**.** *Let* X *be a smooth proper rigid space and assume that the rigid Picard functor is represented by a rigid group* G*. Then, there is a left exact sequence*

$$0 \to G(K)^{tt} \to \mathrm{Pic}(X) \to \mathrm{Pic}(\widetilde{X})$$

where $G(K)^{tt} \subseteq G(K)$ *is the subgroup of elements* x *such that* $x^{n!} \to 1$ *as* $n \to \infty$*.*

3 Vector bundle on pro-finite étale covers

3.1 Pro-finite-étale analytic vector bundles

The first goal of this thesis is to understand which vector bundles on E pullback to the trivial bundle on $\widetilde{E}$ (as before, the base field K is a complete algebraically closed extension of $\mathbb{Q}_p$). To do this, we would first like to expand on Theorem 2.5.4 from above. We shall denote by $\pi : \widetilde{E} \to E$ the natural projections. Sticking to the notation used in [Ati57], we note that

$$\mathrm{Pic}_{E/K}(K) = \bigsqcup_{d \in \mathbb{Z}} \mathcal{E}(1, d),$$

where $\mathrm{Pic}_{E/K}$ is the Picard variety associated to the elliptic curve. Now, consider a map $E \xrightarrow{[n]} E$ for some $n \in \mathbb{Z}_{\geq 1}$. Then, for any line bundle $\mathcal{L} \in \mathrm{Pic}(E)$, $\deg([n]^*\mathcal{L}) = n^2 \deg(\mathcal{L})$. Hence, because the dual isogeny $\widehat{[n]} = [n]$, we have

$$\mathrm{Pic}_{E/K}(K)^{tt} = \mathrm{Pic}^0(E)^{tt} = E(K)^{tt}.$$

In the case of a Tate curve, we have a more precise description of these line bundles.

Proposition 3.1.1. *Let $E_q = {}^{\mathbb{G}_m}/_{q^{\mathbb{Z}}}$ be a Tate curve over some complete algebraically closed extension K of $\mathbb{Q}_p$, where $q \in K$ such that $|q| < 1$ and let Σ be the set of all roots of unity in K. Let,*

$$T := q^{\mathbb{Q}}\Big(\bigcup_{\zeta \in \Sigma} \zeta(1 + m_K\mathcal{O}_K)\Big).$$

Then,

$$E_q(K)^{tt} = {}^{T}/_{q^{\mathbb{Z}}}.$$

Proof. Let $\eta : \mathbb{G}_m \to E_q$ be the quotient map. Firstly, consider the open subset $U = q^{\mathbb{Z}}(1 + m_K\mathcal{O}_K) \subseteq \mathbb{G}_m(K)$. This is the inverse image of an open neighborhood of $e \in E_q(K)$. Therefore, if $x \in \mathbb{G}_m(K)$ is such that its image in the Tate curve lies in $E_q(K)^{tt}$, then $x^{n!} \in U$ for all large enough n. Now, suppose $y \in U$. Then, there is an element $y' \in (1 + m_K\mathcal{O}_K)$ such that they have the same image in $E_q(K)$. Hence, for any $k \in \mathbb{Z}_{\geq 1}$, $\eta(y)^{k\cdot(k+1)\cdots n} \to e$ because $(y')^{k\cdot(k+1)\cdots n} \to 1$ as $n \to \infty$. Therefore, we have that for all $u \in \mathbb{G}_m(K)$, $\eta(u) \in E_q(K)^{tt}$ if and only if $u^{n!} \in U$ for some n.

It is clear that if $x \in T$, then $x^{n!} \in U$ for some n. We would like to show the converse. Suppose $y^{n!} \in U$ for some n. If, $y^{n!} = q^s\theta$, for some $s \in \mathbb{Z}$ and $\theta \in (1 + m_K\mathcal{O}_K)$, we then need to show that every $n!$-th root of θ lies in

$\bigcup_{\zeta\in\Sigma}\zeta(1+\mathfrak{m}_K\mathcal{O}_K)$. This follows from the following commutative diagram of monoids:

$$\begin{tikzcd} \mathcal{O}_K \arrow[r, "(\cdot)^{n!}"] \arrow[d] & \mathcal{O}_K \arrow[d] \\ \mathcal{O}_K/\mathfrak{m} \arrow[r, "(\cdot)^{n!}"] & \mathcal{O}_K/\mathfrak{m} \end{tikzcd}$$

□

Remark 3.1.2. *1. Consider the case* $K=\mathbb{C}_p$. *Then, since its residue field is* $\overline{\mathbb{F}}_p$, *each element in* $\mathcal{O}_K^\times$ *lies in* $\zeta(1+\mathfrak{m}_K\mathcal{O}_K)$ *for some* $\zeta\in\Sigma$. *Moreover, since the value group* $|K^\times|=|q|^{\mathbb{Q}}$, *we end up with* $T=K^\times$, *and hence* $E_q(\mathbb{C}_p)^{tt}=E_q(\mathbb{C}_p)$. *This fact corroborates a more general result in [Heu], which shows the same for any connected smooth proper curve over* $\mathbb{C}_p$.

2. Instead of the pro-finite-étale universal cover, if in 3.1.1 one considers the p-adic pro-finite-étale universal cover

$$E_q^{1/p^\infty}=\varprojlim_{[p^k]}E_q,$$

where k *runs over the whole of* $\mathbb{Z}_{\geq 1}$ *(which is constructed and shown to be a perfectoid space in the paper [Bla+18]), then line bundles which pull back to the trivial line bundle on* E_q^{1/p^∞} *can be described using the topologically p-torsion points of* $E_q(K)$ *(i.e. those points* x, *such that* $x^{p^n}\to e$ *as* $n\to\infty$*), and in this case we get a nice description of the topologically p-torsion points*

$$E_q(K)\langle p^\infty\rangle = {}^{q^{\mathbb{Z}[p^{-1}]}(1+\mathfrak{m}_K\mathcal{O}_K)}/{}_{q^{\mathbb{Z}}},$$

which does not need any additional information about any roots of unity. This is because, using the fact that the Frobenius on a field of characteristic p is an injective morphism, one can show that all p^n*-th roots of unity of* K *lie in* $1+\mathfrak{m}_K\mathcal{O}_K$.

For vector bundles, we first focus on F_r introduced in Theorem 2.1.1, which are the unique (up to isomorphism) degree zero indecomposable vector bundles of rank r which have a global section.

Proposition 3.1.3. *If* $\pi:\widetilde{E}\to E$ *is the natural projection, then* $\pi^*F_r\cong\mathcal{O}_{\widetilde{E}}^{\bigoplus r}$.

Proof. We prove this inductively. Suppose $\pi^*F_r\cong\mathcal{O}_{\widetilde{E}}^{\bigoplus r}$ for some positive r. We have the exact sequence:

$$0\to\mathcal{O}_E\to F_{r+1}\to F_r\to 0.$$

Because we are working with vector bundles (and because completed tensor product is the same as regular tensor product in this case), the sequence still remains exact after we apply π^*. Hence, $\pi^* F_{r+1} \in \operatorname{Ext}^1(\mathcal{O}_{\widetilde{E}}^{\bigoplus r}, \mathcal{O}_{\widetilde{E}}) = H^1(\widetilde{E}, \mathcal{O}_{\widetilde{E}}^{\bigoplus r}) \subseteq H^1_{\text{ét}}(\widetilde{E}, \mathcal{O}_{\widetilde{E}}^{\bigoplus r}) = 0$ because of [Heu22c, Propostion 4.9] and [Heu22b, Propostion 3.9]. Therefore, $\pi^* F_{r+1} \cong \mathcal{O}_{\widetilde{E}}^{\bigoplus r+1}$. Since, $F_1 = \mathcal{O}_E$, the proof is complete. □

In the paper [Ati56], Atiyah proves that for complete varieties, all coherent sheaves have a unique maximal decomposition up to reordering and isomorphisms. A nice consequence of this is that all vector bundles have a unique decomposition (up to reordering) into indecomposable vector bundles. It is not immediately clear if Atiyah's proof can be recreated for vector bundle over 'nice' perfectoid spaces, primarily because Atiyah heavily relies on the fact that the global sections of a coherent sheaf is a finite dimensional vector space over the base field. But, we can however show this for the trivial vector bundles.

Lemma 3.1.4. *Suppose X is a proper connected smooth adic space over K such that its pro-finite-étale universal cover $\widetilde{X}$ is representable by some adic space. Then, the trivial vector bundles on $\widetilde{X}$ have a unique decomposition (up to isomorphisms and reordering) into indecomposable vector bundles (i.e. a decomposition of the form $\mathcal{O}_{\widetilde{X}}^{\bigoplus r}$).*

Proof. Let $T_{(r)}$ denote the trivial bundle of rank r over $\widetilde{X}$. Suppose we have the following two decomposition of the trivial bundle $T_{(r)} \cong \mathcal{O}_{\widetilde{X}}^{\bigoplus r} \cong \bigoplus_{m=1}^{k} V_i$. For each $1 \leq n \leq r$ and $1 \leq m \leq k$, let $i_n : \mathcal{O}_{\widetilde{X}} \to T_{(r)}$ and $i'_m : V_m \to T_{(r)}$ denote the natural inclusions into the respective components, and let $p_n : T_{(r)} \to \mathcal{O}_{\widetilde{X}}$ and $p'_m : T_{(r)} \to V_m$ denote the natural projections into the respective components.

We want to show that $r = k$ and that $V_m \cong \mathcal{O}_{\widetilde{X}}$ for all m. We shall show this inductively. Since line bundles are indecomposable, the result is true for $T_{(1)}$. Suppose it is also true for $T_{(r-1)}$ for some $r \geq 2$.

For each $1 \leq m \leq k$, consider the composition

$$\mathcal{O}_{\widetilde{X}} \xrightarrow{i_1} T_{(r)} \xrightarrow{p'_m} V_m \xrightarrow{i'_m} T_{(r)} \xrightarrow{p_1} \mathcal{O}_{\widetilde{X}}$$

which we shall denote by θ_m. Now, $\theta_m \in \operatorname{Hom}(\mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}) = H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}) = K$ because [Heu22c, Proposition 4.9]. Hence, θ_m is either an isomorphism or zero. Since $\Sigma\theta_m = p_1 \circ i_1 = \mathrm{id}_{\mathcal{O}_{\widetilde{X}}}$, there exists an m such that θ_m is an isomorphism, say $m = 1$. Since the following composition

$$\mathcal{O}_{\widetilde{X}} \xrightarrow{p'_1 \circ i_1} V_1 \xrightarrow{p_1 \circ i'_1} \mathcal{O}_{\widetilde{X}}$$

is an isomorphism, this implies that $\mathcal{O}_{\widetilde{X}}$ is a direct summand of V_1. But, since V_1 is indecomposable, we have that $V_1 \cong \mathcal{O}_{\widetilde{X}}$.

Now, consider the two exact sequences:

$$0 \to V_1 \xrightarrow{i_1'} T_{(r)} \xrightarrow{p'} \bigoplus_{m=2}^{k} V_m \to 0$$

and

$$0 \to T_{(r-1)} \xrightarrow{i} T_{(r)} \xrightarrow{p_1} \mathcal{O}_{\widetilde{X}} \to 0.$$

Because $p_1 \circ i_1'$ is injective, it is easy to see that $p_1' \circ i = 0$ and hence $p' \circ i$ is also injective. In a similar manner, we also have that $p' \circ i_1 = 0$, and therefore $p' \circ i$ is also surjective and hence we have an isomorphism

$$T_{(r-1)} \cong \bigoplus_{m=2}^{k} V_m,$$

which completes the proof. □

Once we know which indecomposable vector bundles pull back to the trivial bundle, Lemma 3.1.4 tells us that a decomposable vector bundle pulls back to the trivial bundle if and only if each of its indecomposable summands pull back to the trivial bundle.

Now that the question has been reduced to indecomposable vector bundles, we would like to further reduce it to only considering degree zero vector bundles. We shall do that in two steps.

Lemma 3.1.5. *Suppose* $V \in \mathcal{E}(r, dr)$ *for some* $r, d \in \mathbb{Z}$ *with* $r \geq 1$. *Then,* π^*V *is trivial only if* $d = 0$.

Proof. Suppose $d \neq 0$. Now, consider the vector bundle F_r on E as defined above. There exists a line bundle $\mathcal{L}$ of degree d on E such that $V \cong F_r \bigotimes \mathcal{L}$ (refer to Remark 2.1.3). Hence, $\pi^*V \cong \mathcal{O}_{\widetilde{E}}^{\bigoplus r} \bigotimes \pi^*\mathcal{L} \cong (\pi^*\mathcal{L})^{\bigoplus r}$. But, since $d \neq 0$, we have that $\pi^*\mathcal{L}$ is not trivial. Since line bundles are indecomposable, Lemma 3.1.4 tells us that π^*V is not trivial. □

Lemma 3.1.6. *Suppose* $V \in \mathcal{E}(r, d)$ *for some* $r, d \in \mathbb{Z}$ *with* $r \geq 1$. *Then,* π^*V *is trivial only if* $d = 0$.

Proof. For any $n, k \in \mathbb{Z}_{\geq 1}$, because the following diagram

$$\begin{tikzcd} E \arrow[r, "{[n]}"] \arrow[d, "{[k]}"'] & E \arrow[d, "{[k]}"] \\ E \arrow[r, "{[n]}"] & E \end{tikzcd}$$

commutes, we get the commutative square

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{[n]} & \widetilde{E} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ E & \xrightarrow{[n]} & E \end{array}$$

Now, suppose $d \neq 0$. Then, there exists a finite sequence of indecomposable vector bundles $V_1, ..., V_k$ of rank $r_1, ..., r_k$ respectively on E, such that the following properties are satisfied:

1. $V_1 = V$.
2. V_{i+1} is an indecomposable summand of $[r_i]^*V_i$ of non-zero degree.
3. $[r_k]^*V_k$ is indecomposable.

Firstly, $\deg([r_k]^*V_k) = r_k^2 \deg(V_k) \neq 0$. Hence, Lemma 3.1.5 tells us that $\pi^*[r_k]^*V_k$ is non-trivial. Therefore, $[r_k]^*\pi^*V_k$ and hence π^*V_k are also non-trivial.

Next, suppose π^*V_{i+1} is non-trivial for some i. Then, Lemma 3.1.4 tells us that $\pi^*[r_i]^*V_i$ is non-trivial. Therefore, just as before, π^*V_i is non-trivial. Recursively following this procedure, we get that $\pi^*V_1 = \pi^*V$ is non-trivial. □

Now that we have the answer for indecomposable vector bundles of non-zero degree, we are reduced to checking those of degree zero. In this case Theorem 2.1.1 from above gives us the desired classification.

Theorem 3.1.7. *The isomorphism* $\mathrm{Pic}^0(X) \to \mathcal{E}(r, 0)$ *described in Theorem 2.1.1 by mapping a line bundle* $\mathcal{L}$ *to the vector bundle* $F_r \bigotimes \mathcal{L}$*, restricts to an isomorphism between those line bundles that pull back to the trivial line bundle on* $\widetilde{E}$*, and those vector bundles of* $\mathcal{E}(r, 0)$ *that pull back to the trivial vector bundle on* $\widetilde{E}$*.*

Proof. Let $V \in \mathcal{E}(r, 0)$. Then, we know that $V \cong F_r \bigotimes \mathcal{L}$ for some unique $\mathcal{L} \in \mathrm{Pic}^0(E)$. Thus, we have $\pi^*V \cong \mathcal{O}_{\widetilde{E}}^{\bigoplus r} \bigotimes \pi^*\mathcal{L} \cong (\pi^*\mathcal{L})^{\bigoplus r}$. Since line bundles are indecomposable, π^*V is trivial if and only if $\pi^*\mathcal{L}$ is trivial by Lemma 3.1.4. □

To stay consistent with notation, we shall denote by $\mathcal{E}(r, 0)^{tt}$ the image of $\mathrm{Pic}^0(E)^{tt}$ in the map described in Theorem 2.1.1.

Theorem 3.1.8. *Let* $\pi : \widetilde{E} \to E$ *be the natural projection. Then, for a vector bundle* V *on* E*,* π^*V *is trivial if and only if there is a unique decomposition (up to isomorphisms and reordering) of* $V = V_1 \bigoplus \cdots \bigoplus V_k$ *such that each* $V_i \in \mathcal{E}(r_i, 0)^{tt}$ *for some* $r_i \in \mathbb{Z}_{\geq 1}$*.*

Proof. As mentioned before, the uniqueness of the decomposition follows from [Ati56, Theorem 3]. Lemma 3.1.4 shows that each indecomposable component must also pull back to the trivial bundle on $\widetilde{E}$. Lastly, each indecomposable component belongs to $\mathcal{E}(r,0)^{tt}$ for some $r \in \mathbb{Z}_{\geq 1}$ follows from Lemma 3.1.6 and Theorem 3.1.7. □

3.2 Vector bundles on $\widetilde{E}$

Suppose for the moment $K = \mathbb{C}_p$. Then, as mentioned previously, [Heu] shows that every line bundle in $\mathrm{Pic}^0(E)$ pulls back to the trivial bundle on $\widetilde{E}$. This implies that every vector bundle in $\mathcal{E}(r,0)$ also pulls back to the trivial bundle on $\widetilde{E}$. If $[e]$ is the identity section of our elliptic curve, then using Theorem 2.1.2 we can describe elements of $\mathcal{E}(r,dr)$ as being those of the form $V \bigotimes \mathcal{O}(d[e])$ for $V \in \mathcal{E}(r,0)$. This has a nice consequence that every vector bundle of $\mathcal{E}(r,dr)$ pulls back to $\pi^*\mathcal{O}(d[e])^{\bigoplus r}$ on $\widetilde{E}$.

Now, we come back to the case where K is a complete algebraically closed extension of $\mathbb{Q}_p$. Another aim is to try and understand the functor:

$$2 - \varinjlim_{[k]} \mathrm{VB}_n(E) \xrightarrow{\pi^*} \mathrm{VB}_n(\widetilde{E}),$$

where $\mathrm{VB}_n(X)$ denotes the category of rank n vector bundles on the space X, and the functor is defined via pullback. We abuse notation from the previous section and call it π^*.

A natural question to ask regarding this functor is whether it is in fact, firstly fully faithful, and secondly essentially surjective. As far as the first part of this question is concerned, the previous section shows that this in fact, need not be fully faithful.

Let E be an elliptic curve over $\mathbb{C}_p$ and $x \in E(\mathbb{C}_p)$ a point of infinite order. Then, there exists a line bundle $\mathcal{L} = \mathcal{O}([x] - [e])$ such that $[n]^*\mathcal{L}$ is non-trivial for all n, but $\pi^*\mathcal{L}$ is in fact trivial. Then, the essential image of $\mathcal{L}$ in the category $2 - \varinjlim_{[k]} \mathrm{VB}_n(E)$ is non-trivial, but its essential image in $\mathrm{VB}_n(\widetilde{E})$ is in fact trivial. Hence we have the following result.

Proposition 3.2.1. *The functor*

$$2 - \varinjlim_{[k]} \mathrm{VB}_n(E) \xrightarrow{\pi^*} \mathrm{VB}_n(\widetilde{E})$$

is not necessarily fully faithful.

We can in fact make a slightly stronger statement. With the help of the previous section, we can describe exactly which objects in the category $2-\varinjlim_{[k]} \mathrm{VB}_n(E)$ go to the trivial object. First, we need to state the following result from the paper [Heu] (the article cited has not been published yet, and Proposition 3.2.2 is being used with the kind permission of the author).

Proposition 3.2.2. *Let* C *be a proper connected curve and*

$$\widetilde{C} = \varprojlim_{C' \to C} C'$$

be its universal pro-finite-étale cover. Then, we have the following exact sequence:

$$0 \to \varinjlim_{C' \to C} \mathrm{Pic}^0(C')^{tt} \to \varinjlim_{C' \to C} \mathrm{Pic}(C') \to \mathrm{Pic}(\widetilde{C}) \to 0,$$

where, throughout this proposition $C' \to C$ *ranges over all connected finite-étale covers of* C*.*

Sketch of the proof. One has to first show that

$$\mathrm{Pic}(\widetilde{C}) = \mathrm{Pic}(\widetilde{C}) \otimes \mathbb{Q}.$$

Then, using the exponential exact sequence:

$$0 \to \mathbb{G}_a \to \mathbb{G}_m \otimes \mathbb{Q} \to \mathbb{G}_m/\mathbb{G}_m^{tt} \to 0,$$

we get the following exact sequence:

$$H^1_{\text{ét}}(\widetilde{C}, \mathbb{G}_a) \to H^1_{\text{ét}}(\widetilde{C}, \mathbb{G}_m) \to H^1_{\text{ét}}(\widetilde{C}, \mathbb{G}_m/\mathbb{G}_m^{tt}) \to H^2_{\text{ét}}(\widetilde{C}, \mathbb{G}_a).$$

One can show that $H^1_{\text{ét}}(\widetilde{C}, \mathbb{G}_a) = H^2_{\text{ét}}(\widetilde{C}, \mathbb{G}_a) = 0$, and hence the natural map

$$H^1_{\text{ét}}(\widetilde{C}, \mathbb{G}_m) \to H^1_{\text{ét}}(\widetilde{C}, \mathbb{G}_m/\mathbb{G}_m^{tt})$$

is an isomorphism. This helps reduce the problem to each finite level C' over C because you have the following result from [Heu21, Proposition 4.11]:

$$H^1_{\text{ét}}(\widetilde{C}, \mathbb{G}_m/\mathbb{G}_m^{tt}) = \varinjlim_{C' \to C} H^1_{\text{ét}}(C', \mathbb{G}_m/\mathbb{G}_m^{tt}).$$

And, at each finite level, we can show the existence of the following exact sequence:

$$0 \to \mathrm{Pic}^0(C')^{tt} \to \mathrm{Pic}^0(C') \to H^1_{\text{ét}}(C', \mathbb{G}_m/\mathbb{G}_m^{tt}) \to \mathbb{Q} \to 0.$$

The result then follows from taking the colimit over all $C' \to C$. $\square$

Since the above proof only relies on the definition of $\widetilde{C}$, the exact same result would work if we consider, in our case of an elliptic curve, the definition $\widetilde{E} = \varprojlim_{[k]} E$. Hence, we have the following exact sequence:

$$0 \to \varinjlim_{[k]} \mathrm{Pic}^0(E)^{tt} \to \varinjlim_{[k]} \mathrm{Pic}(E) \to \mathrm{Pic}(\widetilde{E}) \to 0.$$

We would firstly like to generalise the 'left part' of this exact sequence. For this, we need to state a small lemma which we will need.

Lemma 3.2.3. *Let C be a proper connected curve and $\phi : C' \to C$ a finite-étale cover with C' connected. Then, the diagram*

$$\begin{tikzcd} \mathrm{Pic}(C)^{tt} \arrow[r, hook] \arrow[d, "\phi^*"'] & \mathrm{Pic}(C) \arrow[d, "\phi^*"] \\ \mathrm{Pic}(C')^{tt} \arrow[r, hook] & \mathrm{Pic}(C') \end{tikzcd}$$

is Cartesian.

Proof. Suppose $\mathcal{L} \in \mathrm{Pic}(C)$ such that $\phi^*\mathcal{L} \in \mathrm{Pic}(C')^{tt}$. Now, Proposition 3.2.2 shows that $\pi_C^*\mathcal{L}$ is trivial, where $\pi_C : \widetilde{C} \to C$ is the natural projection. Theorem 2.5.4 tell us that this implies that $\mathcal{L} \in \mathrm{Pic}(C)^{tt}$. $\square$

Remark 3.2.4. *For our case of elliptic curves and in the cases where the finite-étale map is of the form $E \xrightarrow{[n]} E$, we have another proof. Since the dual isogeny $\widehat{[n]} = [n]$, we can instead look at the Jacobian of our elliptic curve (which is isomorphic to E) $\mathrm{Pic}^0_{E/K}(K) \xrightarrow{(\cdot)^{\bigotimes n}} \mathrm{Pic}^0_{E/K}(K)$. Now, [Heu22a, Corollary 3.8], shows that $E \cong \mathrm{Pic}^0_{E/K}$ has a neighbourhood basis of the identity $(U_k)_{k \in \mathbb{Z}_{\geq 1}}$, consisting of open subgroups $U_k \subseteq \mathrm{Pic}^0_{E/K}$. This system of neighbourhoods easily helps us verify that for any $x \in \mathrm{Pic}^0_{E/K}(K)$, we have $x \in (\mathrm{Pic}^0_{E/K}(K))^{tt}$ if and only if $x^m \in (\mathrm{Pic}^0_{E/K}(K))^{tt}$ for any $m \in \mathbb{Z}_{\geq 1}$.*

Now, for an elliptic curve we shall denote by $\mathrm{VB}_n(E)^{tt}$ the subcategory of $\mathrm{VB}_n(E)$ consisting of those those vector bundles described in Theorem 3.1.8. Then, Lemma 3.2.3 shows us that for any $k \in \mathbb{Z}_{\geq 1}$, the functor $\mathrm{VB}_n(E)^{tt} \xrightarrow{[k]^*} \mathrm{VB}_n(E)^{tt}$ is well defined. To show this, we first fix some notation. For any $m \in \mathbb{Z}_{\geq 1}$, let $\pi_m : \widetilde{E} \to E$ such that the following diagram commutes:

$$\begin{tikzcd} \widetilde{E} \arrow[rr, "\pi_m"] \arrow[dr, "\pi"'] & & E \arrow[dl, "{[m]}"] \\ & E & \end{tikzcd}$$

Lemma 3.2.5. *For any $k \in \mathbb{Z}_{\geq 1}$ and $V \in \mathrm{VB}_n(E)^{tt}$, $[k]^*V$ is also an object of $\mathrm{VB}_n(E)^{tt}$.*

Proof. Combining Theorem 3.1.7 and Theorem 3.1.8, we only need to show this result for $V = F_n$ for all $n \in \mathbb{Z}_{\geq 1}$. On the contrary, let us assume $[k]^*F_n$ has an indecomposable component, say $V' \subseteq [k]^*F_n$, which is of non-zero degree. Without loss of generality, we can assume (by potentially taking a large enough k - in a manner similar to the proof of Lemma 3.1.6) that $\mathrm{rank}(V') \big| \deg(V')$. Hence, $V' = F_{n'} \bigotimes \mathcal{L}$ for some $n' \geq 1$ and $\mathcal{L} \in \mathrm{Pic}(E)$ of non-zero degree. Now, combining Proposition 3.2.2 and Lemma 3.2.3, we have that $\pi_k^*\mathcal{L}$ is non-trivial, and for exactly the same reasons as in Proposition 3.1.3, $\pi_k^*F_{n'} = \mathcal{O}_{\widetilde{E}}^{\bigoplus n'}$. Hence, Lemma 3.1.4 tells us that $\pi_k^*[k]^*F_n = \pi^*F_n$ is non-trivial. This is a contradiction. Therefore,

$$[k]^*F_n \cong \bigoplus_{i=1}^{m} F_{n_i} \otimes \mathcal{L}_i$$

for $n_i \geq 1$ and $\mathcal{L}_i \in \mathrm{Pic}^0(E)$.

Hence, once again combining Proposition 3.2.2 and Lemma 3.2.3, we have that $\mathcal{L}_i \in \mathrm{Pic}^0(E)^{tt}$. Therefore, $[k]^*F_n \in \mathrm{VB}_n(E)^{tt}$. □

Using a very similar strategy to previous theorems' proofs, we can show the following:

Theorem 3.2.6. *The essential image of the functor*

$$2 - \varinjlim_{[k]} \mathrm{VB}_n(E)^{tt} \to 2 - \varinjlim_{[k]} \mathrm{VB}_n(E)$$

are exactly those objects which go to the trivial bundle on the pro-finite-étale cover $\widetilde{E}$.

Proof. For some infinite subset $S \subseteq \mathbb{Z}_{\geq 1}$, such that for all $k \in S$, $k\mathbb{Z}_{\geq 1} \subseteq S$, let $\{V_k\}_{k \in S}$ be a family of vector bundles on E such that for each $k \in S$ and $n \in \mathbb{Z}_{\geq 1}$, $V_{kn} \cong [n]^*V_k$.

Suppose there exists some $k \in S$ such that V_k has an indecomposable component, say $V_k' \subseteq V_k$ which is of non zero degree. Then, just as in the previous lemma, this would imply that $\pi_k^*V_k$ is non-trivial.

Now, suppose each V_k decomposes into indecomposable vector bundles of degree 0 for all $k \in S$. Now, let $l \in S$. Suppose, $V \subseteq V_l$ is one of these indecomposable components. Then, $V \cong F_r \bigotimes \mathcal{L}$ for some $r \geq 1$ and $\mathcal{L} \in \mathrm{Pic}^0(E)$. Now, combining Proposition 3.2.2 and Lemma 3.2.3, we have that π_l^*V is trivial if and only if $\mathcal{L} \in \mathrm{Pic}(E)^{tt}$. Therefore, Lemma 3.1.4 tells us that $\pi_l^*V_l$ is trivial if and only $V_l \in \mathrm{VB}_m(E)^{tt}$ for $m = \mathrm{rank}(V_l)$. □

The next obvious step is to try and check if every vector bundle on $\widetilde{E}$ comes from some finite level. While we will not be able to definitively prove this, we will look at a few results in this direction. If one observes the proof of Proposition 3.2.2, then it can be seen that the idea is to somehow compare the cohomology group $H^1(\widetilde{E}, \mathbb{G}_m)$ to cohomologies at each finite level.

We would ideally like to do something similar for vector bundles. Rather than looking at the sheaf of groups GL_n, it would be better to look at the sheaf $\mathrm{GL}_n(\mathcal{O}^+)$ which classifies the integral models of vector bundles of rank n. Hopefully, if we can show that these locally free $\mathcal{O}^+$-modules come from finite levels, then we would like to show that we have a similar result for vector bundles.

First and foremost, for any $s \in \mathbb{Z}_{\geq 1}$, we have the following map

$$\mathrm{GL}_n(\mathcal{O}^+) \to \mathrm{GL}_n(\mathcal{O}^+/p^s\mathfrak{m}_K).$$

We would like to first express $\mathrm{H}^1(\widetilde{E}, \mathrm{GL}_n(\mathcal{O}^+))$ in terms of $\mathrm{H}^1(\widetilde{E}, \mathrm{GL}_n(\mathcal{O}^+/p^s\mathfrak{m}_K))$.

Definition 3.2.7. *A* ***generalised representation*** *on* $\widetilde{E}$ *is a system* $(M_k)_{k\in\mathbb{Z}_{\geq 1}}$ *of finite locally free* $\mathcal{O}^+/p^k$*-modules* M_k *on* $\widetilde{E}_{\text{ét}}$ *along with isomorphisms* $M_{k+1}/p^k \xrightarrow{\cong} M_k$ *for all* k.

We now have the following two results:

Theorem 3.2.8 ([Heu22a], Theorem 2.3)**.** *The morphism of sites* $\nu : \widetilde{E}_\nu \to \widetilde{E}_{\text{ét}}$ *induces a natural equivalence of categories:*

$$\begin{array}{c}\{\textit{finite locally free } \mathcal{O}^+\textit{-modules on } \widetilde{E}_\nu\}\\ \updownarrow \\ \{\textit{generalised representations on } \widetilde{E}\}\end{array}$$

Theorem 3.2.9 ([Heu22a], Theorem 2.21)**.** *The categories of finite locally free* $\mathcal{O}^+$*-modules on* $\widetilde{E}_{\text{ét}}$ *and* $\widetilde{E}_\nu$ *are equivalent.*

One observes that the proof of Theorem 3.2.8 still works if we replace generalised representations by locally free $\mathcal{O}^+/p^s\mathfrak{m}_K$-modules. This is because, as mentioned in [Heu22a, Lemma 2.6], we have an isomorphism

$$\mathrm{H}^i_{\text{ét}}(\widetilde{E}, \mathrm{GL}_n(\mathcal{O}^+/p^s\mathfrak{m}_K)) \cong \mathrm{H}^i_\nu(\widetilde{E}, \mathrm{GL}_n(\mathcal{O}^+/p^s\mathfrak{m}_K))$$

for both $i = 0$ and 1. This helps show that the functor is both well defined and fully faithful. Essential surjectivity follows from [Heu22a, Lemma 2.24]. Then, as a consequence of the theorems mentioned above, we have the following isomorphism:

$$\mathrm{H}^1_{\text{ét}}(\widetilde{E}, \mathrm{GL}_n(\mathcal{O}^+)) \cong \varprojlim_{s\in\mathbb{Z}_{\geq 1}} \mathrm{H}^1_{\text{ét}}(\widetilde{E}, \mathrm{GL}_n(\mathcal{O}^+/p^s\mathfrak{m}_K)).$$

Moreover, [Heu22a, Lemma 2.16] shows us that:

$$\mathrm{H}^1_{\text{ét}}(\widetilde{E}, \mathrm{GL}_n(\mathcal{O}^+/p^s\mathfrak{m}_K)) = \varinjlim_{[k]} \mathrm{H}^1_{\text{ét}}(E, \mathrm{GL}_n(\mathcal{O}^+/p^s\mathfrak{m}_K)).$$

So as to work in a manner similar to the $n = 1$ case, it would be nice if we had

$$\mathrm{H}^1_{\text{ét}}(\widetilde{E}, \mathrm{GL}_n(\mathcal{O}^+/p^{s+1}\mathfrak{m}_K)) \cong \mathrm{H}^1_{\text{ét}}(\widetilde{E}, \mathrm{GL}_n(\mathcal{O}^+/p^s\mathfrak{m}_K)).$$

But, if we examine the short exact sequence

$$0 \to \mathrm{M}_n({}^{\mathfrak{m}_K\mathcal{O}^+}/_{p\mathfrak{m}_K}) \to \mathrm{GL}_n({}^{\mathcal{O}^+}/_{p^{s+1}\mathfrak{m}_K}) \to \mathrm{GL}_n({}^{\mathcal{O}^+}/_{p^s\mathfrak{m}_K}) \to 0,$$

then we have the following long exact sequence of pointed sets

$$\begin{tikzcd} H^1_{\text{ét}}(\widetilde{E}, \mathrm{M}_n({}^{\mathfrak{m}_K\mathcal{O}^+}/_{p\mathfrak{m}_K})) \arrow[r] & H^1_{\text{ét}}(\widetilde{E}, \mathrm{GL}_n({}^{\mathcal{O}^+}/_{p^{s+1}\mathfrak{m}_K})) \arrow[dl] \\ H^1_{\text{ét}}(\widetilde{E}, \mathrm{GL}_n({}^{\mathcal{O}^+}/_{p^s\mathfrak{m}_K})) \arrow[r] & H^2_{\text{ét}}(\widetilde{E}, \mathrm{M}_n({}^{\mathfrak{m}_K\mathcal{O}^+}/_{p\mathfrak{m}_K})). \end{tikzcd}$$

For $i = 1$, [Heu22b, Proposition 3.9] shows us that in fact we have $H^1_{\text{ét}}(\widetilde{E}, \mathcal{O}^{+a}/p) = 0$, and [Heu] shows the same for $i = 2$ as follows: as before [Heu22a, Lemma 2.16] tells us that

$$H^2_{\text{ét}}(\widetilde{E}, \mathcal{O}^+/p) = \varinjlim_{[k]} H^2_{\text{ét}}(E, \mathcal{O}^+/p),$$

and [Sch13, Proposition 5.1] shows us that we have an isomorphism of almost K°-modules:

$$H^2_{\text{ét}}(E, \mathcal{O}^{+a}/p) \cong H^2_{\text{ét}}(E, \mathbb{Z}/p\mathbb{Z}) \otimes K^{\circ a}/p.$$

By Poincaré duality, we have $H^2_{\text{ét}}(E, \mathbb{Z}/p\mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}$. One can understand the action of $[k]^*$ on $H^2_{\text{ét}}(E, \mathbb{Z}/p\mathbb{Z})$ using the Kummer sequence. Then, taking the colimit, we get $\varinjlim_{[k]} H^2_{\text{ét}}(E, \mathbb{Z}/p\mathbb{Z}) = 0$, which gives us the following isomorphism of almost K°-modules:

$$H^1_{\text{ét}}(\widetilde{E}, \mathcal{O}^{+a}/p) = H^2_{\text{ét}}(\widetilde{E}, \mathcal{O}^{+a}/p) = 0.$$

Now, since $\widetilde{E}$ is quasi-compact and $\mathfrak{m}_K$ is a flat K°-module, we have that

$$H^i_{\text{ét}}(\widetilde{E}, \mathfrak{m}_K\mathcal{O}^+/p\mathfrak{m}_K) = H^i_{\text{ét}}(\widetilde{E}, \mathcal{O}^+/p) \otimes \mathfrak{m}_K = 0$$

for both $i = 1$ and 2. Therefore, we have

$$H^1_{\text{ét}}(\widetilde{E}, \mathrm{M}_n({}^{\mathfrak{m}_K\mathcal{O}^+}/_{p\mathfrak{m}_K})) = H^2_{\text{ét}}(\widetilde{E}, \mathrm{M}_n({}^{\mathfrak{m}_K\mathcal{O}^+}/_{p\mathfrak{m}_K})) = 0.$$

But, this unfortunately does not imply that

$$H^1_{\text{ét}}(\widetilde{E}, \mathrm{GL}_n({}^{\mathcal{O}^+}/_{p^{s+1}\mathfrak{m}_K})) \to H^1_{\text{ét}}(\widetilde{E}, \mathrm{GL}_n({}^{\mathcal{O}^+}/_{p^s\mathfrak{m}_K}))$$

is injective.

But, it might be possible to fix some of these issues in the case of good reduction and hence we leave that as a conjecture which hopefully paves the way for future work.

Conjecture 3.2.10. *If* E *is an elliptic curve over* K *with good reduction, then the functor*

$$2-\varinjlim_{[k]} \mathrm{VB}_n(\mathsf{E}) \xrightarrow{\pi^*} \mathrm{VB}_n(\widetilde{\mathsf{E}})$$

is essentially surjective.

The idea is to consider the elliptic curve's reduction mod $\mathfrak{m}_K$, and since this is also an elliptic curve over an algebraically closed field, see if we can somehow use Atiyah's classification to get additional information about $\mathrm{H}^1_{\text{ét}}(\widetilde{\mathsf{E}}, \mathrm{GL}_n(\mathcal{O}^+))$.

References

[Ati56] Michael F Atiyah. "On the Krull-Schmidt theorem with application to sheaves". In: *Bulletin de la Société mathématique de France* 84 (1956), pp. 307–317.

[Ati57] Michael Francis Atiyah. "Vector bundles over an elliptic curve". In: *Proceedings of the London Mathematical Society* 3.1 (1957), pp. 414–452.

[BGR84] Siegfried Bosch, Ulrich Güntzer, and Reinhold Remmert. *Non-archimedean analysis*. Vol. 261. Springer Berlin, 1984.

[Bla+18] Clifford Blakestad et al. "Perfectoid covers of abelian varieties". In: *arXiv preprint arXiv:1804.04455* (2018).

[Bos14] Siegfried Bosch. *Lectures on formal and rigid geometry*. Vol. 2105. Springer, 2014.

[Han15] David Hansen. *Perfectoid universal covers of curves*. https://arithmetica.wordpress.com/2015/09/27/perfectoid-universal-covers-of-curves/. 2015.

[Heu] Ben Heuer. "Picard groups of perfectoid covers: Néron-Severi groups". Article not yet publicly available.

[Heu21] Ben Heuer. "Diamantine Picard functors of rigid spaces". In: *arXiv preprint arXiv:2103.16557* (2021).

[Heu22a] Ben Heuer. "G-torsors on perfectoid spaces". In: *arXiv preprint arXiv:2207.07623* (2022).

[Heu22b] Ben Heuer. "A geometric p-adic Simpson correspondence in rank one". In: *arXiv preprint arXiv:2207.13657* (2022).

[Heu22c] Ben Heuer. "Line bundles on rigid spaces in the v-topology". In: *Forum of Mathematics, Sigma*. Vol. 10. Cambridge University Press. 2022, e82.

[HMW22] Ben Heuer, Lucas Mann, and Annette Werner. "The p-adic Corlette–Simpson correspondence for abeloids". In: *Mathematische Annalen* (2022), pp. 1–38.

[Hub13] Roland Huber. *Étale cohomology of rigid analytic varieties and adic spaces*. Vol. 30. Springer, 2013.

[Hub93] Roland Huber. "Continuous valuations". In: *Math. Z* 212.3 (1993), pp. 455–477.

[Hub94] Roland Huber. "A generalization of formal schemes and rigid analytic varieties". In: *Mathematische Zeitschrift* 217.1 (1994), pp. 513–551.

[Ray74] Michel Raynaud. "Géométrie analytique rigide d'apres Tate, Kiehl". In: *Table ronde d'analyse non archimédienne (Paris, 1972)* (1974), pp. 319–327.

[Sch12] Peter Scholze. "Perfectoid spaces". In: *Publications mathématiques de l'IHÉS* 116.1 (2012), pp. 245–313.

[Sch13] Peter Scholze. "p-adic Hodge theory for rigid-analytic varieties". In: *Forum of Mathematics, Pi*. Vol. 1. Cambridge University Press. 2013, e1.

[Sch17] Peter Scholze. "Étale cohomology of diamonds". In: *arXiv preprint arXiv:1709.07343* (2017).

[Ser56] Jean-Pierre Serre. "Géométrie algébrique et géométrie analytique". In: *Annales de l'institut Fourier*. Vol. 6. 1956, pp. 1–42.

[Sil09] Joseph H Silverman. *The arithmetic of elliptic curves*. Vol. 106. Springer, 2009.

[Tat71] J. Tate. "Rigid Analytic Spaces." In: *Inventiones mathematicae* 12 (1971), pp. 257–289. URL: http://eudml.org/doc/142078.