TALK 1

Shimura Varieties

§1. Hermitian symmetric domains.

We, number theorists often like to define symmetric spaces in the form

$$G(\mathbb{R})^+ / K_\infty$$

But, this is an incorrect definition!!

Definition: A *symmetric space* is a connected Riemanian manifold (M, g) which is homogenous and $\forall p \in M$, $\exists$ an involution s_p of M, with p as an isolated fixed point.

For a Riemmanian manifold (M, g),

$\mathrm{Is}(M,g)$ is a Lie group

[Cartan (Jr.), Cartan (Sr.), Myers - Steenrod]

Lemma 1: Let (M,g) be a symmetric space, let $p \in M$. Then, the subgroup $K_p \subseteq \mathrm{Is}(M,g)^+$ fixing p is compact, and

$$\begin{aligned} \mathrm{Is}(M,g)^+/K_p &\longrightarrow M \\ a \cdot K_p &\longmapsto a \cdot p \end{aligned}$$

is an isomorphism of smooth manifolds.

$\mathrm{Is}(M,g)^+$ need not be the $\mathbb{R}$-points of an algebraic group.

invariance of g under some "almost-complex" structure on the tangent spaces

Negative curvature

Proposition 2: Let (M, g) be a hermitian symmetric domain, and let $\mathfrak{g} = \mathrm{Lie}(\mathrm{Is}(M^\infty, g)^+)$. Then, there exist a unique connected algebraic subgroup $G \subseteq GL(\mathfrak{g})$ such that

$$G(\mathbb{R})^+ = \mathrm{Is}(M^\infty, g)^+ \qquad \text{(Inside } GL(\mathfrak{g})),$$

and

$$G(\mathbb{R})^+ = G(\mathbb{R}) \cap \mathrm{Is}(M^\infty, g) \quad \text{(Inside } GL(\mathfrak{g})).$$

Important Note: The G above is adjoint.

A remark on Hermitian: Shimura varieties are quotients of certain Riemannian manifolds which have a complex ~~algebraic~~ structure. Since $\mathcal{H}^3$ has odd real dimension, it cannot be Hermitian and will not give us the desired complex algebraic varieties.

§2. Short interlude: Hodge structures

Definition 3: A real Hodge structure is a pair (V, h), where $V/\mathbb{R}$ is a finite dim'l vector space, and h is a decomposition:

$$V \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{p,q}$.

Remark: The type of a Hodge structure is a subset $\{\cdots\} \subseteq \mathbb{Z}^2$ such that $V^{p,q} = 0$ for all $(p,q) \in \mathbb{Z}^2 \setminus \{\cdots\}$. We say that a Hodge structure is pure of wt. n if $V^{p,q} \neq 0$ implies $p+q = n$. In general, any H.S. can be written as a direct sum

weight decomposition.

$$V = \bigoplus_{n \in \mathbb{Z}} V_n$$

of H.S. V_n which are pure of wt. n.

We can analogously define $\mathbb{Q}$- / $\mathbb{Z}$- Hodge structures, but then the weight decomposition should be defined over $\mathbb{Q}$.

<u>Important example</u>:

$X/_{\mathbb{C}}$ is a non-singular, then the Hodge decomposition

$$H^n(X, \mathbb{Q}) \otimes \mathbb{C} \cong \bigoplus_{p+q=n} H^p(X, \Omega^q)$$

give $H^n(X, \mathbb{Q})$ a rational Hodge structure, pure of weight n.

More natural in some ways, because spliting may not be canonical

Given a H.S. of wt. n, we can define a filteration:

$$\cdots \supseteq F^p V \supseteq F^{p+1} V \supseteq \cdots \quad ; \quad F^p V = \bigoplus_{r \geq p} V^{r,s}$$

We have an equivalence of categories:

The notation (V,h) denotes the vector space & the map $h: \mathbb{S} \to GL(V)$

$$\left\{ \text{Finite dim'l } \mathbb{R}\text{-rep}^{n} \text{ of } \mathbb{S} = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_m \right\}$$

$z \in \mathbb{C}^\times$, $v \in V^{p,q}$

$z \cdot v = z^{-p} \bar{z}^{-q} \cdot v$

$$\Big\updownarrow \cong$$

$$\left\{ \text{real Hodge structures} \right\}$$

If we restrict to only wt-n $\quad \Big\updownarrow \cong$

$$\left\{ \text{Hodge filberations } (V, F^{\bullet}) \right\}$$

Check: Direct sums and tensor products of H.S. makes sense!

Notation: $\mathbb{R}(n)$ (similarly $\mathbb{Z}(1)$ & $\mathbb{Q}(1)$) are the unique one dim'l H.S. of type $(-n,-n)$.

<u>Definition 4</u>: Suppose (V, h) is of wt. n.

A <u>polarization</u> of (V, h) is a morphism

$\psi : V \otimes V \longrightarrow \mathbb{R}(-n)$ such that

$$\psi_{h(i)}(u, v) := \psi\big(u, h(i)\, v\big)$$

is symmetric and positive definite.

<u>Definition 5</u>: Let S be a connected complex manifold, $V/\mathbb{R}$ a vector space and $n \in \mathbb{Z}$. A family of Hodge structure $\{h_s\}_{s \in S}$ of wt. n is called a <u>variantion</u> if:

(i) CONTINUITY: For all $p, q \in \mathbb{Z}$,

$$\dim V_{h_s}^{p,q} = \dim V_{h_{s'}}^{p,q} = d(p,q) \quad \forall s, s' \in S,$$

and the map

$$\begin{aligned} S &\longrightarrow Gr_{d(p,q)}(V \otimes \mathbb{C}) \\ s &\longmapsto V_{h_s}^{p,q} \end{aligned}$$

is continuous.

(ii) HOLOMORPHICITY: If $d = (\ldots, d(p), \ldots)$, where $d(p) = \sum_{r \geq p} d(r, n-r)$, the map

$$\begin{aligned} S &\xrightarrow{\varphi} G_d(V \otimes \mathbb{C} \\ s &\longmapsto \cdots \subseteq F_{h_s}^{p+1} V \subseteq F_{h_s}^{p} V \subseteq \cdots \end{aligned}$$

is holomorphic.

(iii) GRIFFITHS TRANSVERSALIT: The image of the maps

Heuristically, it says that it takes something of "degree" p to "degree" p-1 under differentiation. Can be understood better using local systems & Gauss-Manin connection:
$\nabla : F^p V \longrightarrow F^{p-1} V \otimes \Omega_S \subseteq V \otimes \Omega_S$

$$d\varphi_s : Tgt_s S \longrightarrow Tgt_{F^\bullet_{h_s}}(G_d(V \otimes \mathbb{C})) \subseteq \bigoplus_p \mathrm{Hom}\left(F^p_{h_s}, V \otimes \mathbb{C}/F^p_{h_s}\right)$$

is contained in $\bigoplus_p \mathrm{Hom}\left(F^p_{h_s}, F^{p-1}_{h_s}/F^p_{h_s}\right)$.

—— x ——

It is said to be a variantian of polarizable Hodge structures if there exists a map $\psi : V \otimes V \longrightarrow \mathbb{R}$ such that for each $s \in S$, ψ_{h_s} is a polarization

Example: Let $\pi : \mathcal{E} \longrightarrow S$ be a morphism of non-singular algebraic varieties$/\mathbb{C}$. The Hodge decomposition on $\{H^n(\mathcal{E}_s, \mathbb{Q})\}_{s \in S}$ gives a variation of polarizable Hodge structures on $S(\mathbb{C})$.

<u>Theorem 6</u>: If $S(V, d, T)$ is the space of all Hodge structure on V, with $d : \mathbb{Z} \times \mathbb{Z} \to \mathbb{N}$ determining the dimensions, & T is a family of morphisms $t : V^{\otimes r} \to \mathbb{R}(-nr/2)$ such that $t_0 \in T$ is a polarizarization, then any connected component $S^+ \subseteq S(V, d, T)$:

$$\{h_s\}_{s \in S^+} \text{ is a variation}$$

$$\Downarrow$$

Converse is true if S^+ is irreducible: Not a product of lower dimensional Her. sym. spaces.

S^+ has a unique complex structure for which it is a Hermitian symmetric domain.

§3. Examples

My go-to example for Riemmanian manifolds are the Hyperbolic spaces $\mathcal{H}^g$

But, as we have mentioned before, if g is odd, we have no complex structure on it.

Let's rewrite $\mathcal{H}^{2g}$ in some usable form.

$$\mathcal{H}^{2g} = \left\{ X + iY \in M_{g\times g}(\mathbb{C}) \;\middle|\; \begin{array}{c} X, Y \in M_{g\times g}(\mathbb{R}) \\ {}^tX = X \ \& \ {}^tY = Y \\ Y > 0 \end{array} \right\}$$

$$\text{Now, } Sp_{2g}(\mathbb{R}) = \left\{ M \in GL_{2g}(\mathbb{R}) \;\middle|\; {}^tM \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} M = \begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix} \right\}$$

Transitively

$\mathcal{H}^{2g}$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ+B)(CZ+D)^{-1}.$$

when $g = 1 \rightsquigarrow Sp_2 = SL_2$ & $\mathcal{H}^{2g} = \mathcal{H}$

Notation: $\mathcal{H}_g^+ = \mathcal{H}^{2g}$

Siegel upper half plane

$$\mathcal{H}_g^- = \left\{ X + iY \in M_{g\times g}(\mathbb{C}) \;\middle|\; \begin{matrix} X, Y \in M_{g\times g}(\mathbb{R}) \\ {}^tX = X \ \& \ {}^tY = Y \\ Y < 0 \end{matrix} \right\}$$

$$\mathcal{H}_g^{\pm} = \mathcal{H}_g^+ \sqcup \mathcal{H}_g^-$$

Siegel upper & lower half-plane

$\begin{pmatrix} 0 & -I_g \\ I_g & 0 \end{pmatrix}$ is an involution on $\mathcal{H}_g$

with iI_g the only fixed point

$\Rightarrow$ $\mathcal{H}_g^+$ is a symmetric space.

$\leadsto$ We shall take it as gospel that it infact a hermitian symmetric domain

$\leadsto$ The real algebraic group associated to $\mathcal{H}_g$ is $(Sp_{2g})^{ad} = \overset{\text{probably non-standard notation}}{PGSp_{2g}} = (GSp_{2g})^{ad}$ $\quad (G^{der})^{ad} = G^{ad}$

Now, $GSp_{2g}(\mathbb{R}) \curvearrowright \mathcal{H}_g^{\pm}$ (fun exercise!)

Another, not so hard exercise:

Find $K \leq Sp_{2g}(\mathbb{R})$ so that

$$\mathcal{H}_g = Sp_{2g}(\mathbb{R}) / K$$

§4. But, where are the varieties

Notation: In this section (only) if D is a Hermitian symmetric domain, then G will be the associated unique adjoint real algebraic group from Proposition 2.

Recall:
* We discussed Hermitian symmetric domains via examples.
* We discussed the existence of a unique adjoint group associated to Hermitian symmetric domains.
* We discussed the relationship between Hermitian symmetric domains & variation of polarizable Hodge structures.

Theorem 7 (Bailey-Borel, Borel): Let ${}_{\Gamma}\backslash D$ be the quotient of a hermitian symmetric domain by a torsion free subgroup $\Gamma \leq G(\mathbb{R})^{+}$. If Γ is "small enough", then ${}_{\Gamma}\backslash D$ has a canonical realization as a Zariski-open subset of a projective variety ${}_{\Gamma}\backslash D^*$. In particular, it has a canonical structure of a algebraic variety which happens to be unique. (everything over $\mathbb{C}$)

${}_{\Gamma}\backslash D^*$ – minimal compactification / standard compt. / Satake-Bailey-Borel compt. / Bailey-Borel compt.

Historically, Shimura & his students Miyake & wanted to construct varieties $X_{(D,\Gamma)}$ over number fields such that

$$X_{(D,\Gamma)} \otimes \mathbb{C} \cong \Gamma \backslash D$$

for certain Ds & Γs.

But Deligne realized the theory can be formalised a lot better using the G's

Theorem 8: Let D & G be as before. For each $p \in D$, there morphism

$$h_p : \mathbb{S} \longrightarrow G$$

such that $h_p(z)$ fixes p & acts on $Tgt_p(D)$ as multiplication by $z/\bar{z}$.

Additionally:

(a) The Hodge structure on $\mathrm{Lie}(G)$ defined by $\mathrm{Ad} \circ h_p$ is of type

$$\{(-1,1), (0,0), (1,-1)\}.$$

Recall that G is adjoint

(b) $\mathrm{ad}(h_p(i))$ is a Cartan involution of G.

(c) G has no simple factor H on which the projection of h_p is trivial.

The converse of the statement is also true!

§5. (Connected) Shimura datum & (connected) Shimura varieties

Notation: $G/_{\mathbb{Q}}$ will be a reductive group

If $X = \{ \mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \}_h$ is some set, then

$$\overline{X} = \{ \mathbb{S} \xrightarrow{h} G_{\mathbb{R}} \longrightarrow G^{ad}_{\mathbb{R}} \}_h .$$

Let (G, X) be a pair are above. We give names for the following properties:

SV1: for all $h \in X$, the Hodge structure on $\mathrm{Lie}\, G_{\mathbb{R}}$ defined by $\mathrm{Ad} \circ h$ is of type

$$\{(-1,1), (0,0), (1,-1)\};$$

SV2: for all $h \in X$, $\mathrm{ad}(h(i))$ is a Cartan involution of $G^{ad}_{\mathbb{R}}$;

SV3: G^{ad} has no $\mathbb{Q}$-factor on which the projection of h is trivial

<u>Definition 9</u>: A <u>Shimura datum</u> (resp. <u>connected Shimura datum</u>) is a pair (G, X) consisting of a reductive group (resp. semisimple group) over $\mathbb{Q}$ and a $G(\mathbb{R})$-conjugacy class (resp. $G^{ad}(\mathbb{R})^{+}$-conjugacy class) of homomorphisms $h : \mathbb{S} \longrightarrow G_{\mathbb{R}}$ (resp. $h : \mathbb{S} \longrightarrow G^{ad}_{\mathbb{R}}$) satisfying SV1, SV2 and SV3.

<u>Example</u>: Firstly, we have a diagram

$$\begin{tikzcd}
 & GSp_{2g}^{der} = Sp_{2g} \arrow[d] \arrow[dr] & \\
\mathbb{G}_m \arrow[r] \arrow[dr] & GSp_{2g} \arrow[r] \arrow[d] & GSp_{2g}^{ad} \\
 & \mathbb{G}_m &
\end{tikzcd}$$

$(Sp_{2g}, \mathcal{H}_g^{+})$ - connected Shimura datum

$(GSp_{2g}, \mathcal{H}_g^{\pm})$ - Shimura datum

Reminder: A congruence subgroup of G is one that contains $\Gamma(N)$ for some embedding $G \hookrightarrow GL_N$.

<u>Definition 10</u>: Let (G, D) be a connected Shimura datum. A <u>connected Shimura variety</u> of "small enough" level $\Gamma \leq G^{ad}(\mathbb{Q})^+$, which contains a congruence subgroup of $G(\mathbb{Q})^+$ whose image image in $\mathrm{Is}(D^\infty, g)$ is torsion free, is a complex algebraic variety of the form $\Gamma \backslash D$.

The inverse system of such algebraic varieties, denoted by $Sh^\circ(G,D)$ is the <u>Shimura variety</u> attached to (G,D).

<u>Theorem 11</u>: If $_{\Gamma}\backslash^{D}$ is a connected Shimura variety with some "good assumptions" on G, where there is some compact open subgroup $K \subseteq G(\mathbb{A}_f)$. Then, there is a natural bijection:

$$_{\Gamma}\backslash^{D} \simeq {}_{G(\mathbb{Q})}\backslash^{D \times G(\mathbb{A}_f)}/_{K}$$

The above isomorphism agrees with varying $K \subseteq G(\mathbb{A}_f)$

<u>Example</u>: The most obvious examples are modular cures like:

$${}_{\Gamma(N)}\backslash^{\mathcal{H}} \quad , \quad {}_{\Gamma_0(p^n)}\backslash^{\mathcal{H}} \quad , \text{ etc.}$$

Definition 12: Given a Shimura datum (G, X) & $K \subseteq G(\mathbb{A}_f)$ compact open. A Shimura variety of level K is a complex algebraic variety $Sh_K(G,X)$ such that

$$Sh_K(G,X)(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K .$$

A Shimura variety attached to (G,X) is the inverse system:

$$Sh(G,X) = \varprojlim Sh_K(G,X)$$

Under "nice assumptions" on G, we get:

$$G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K \simeq \bigsqcup_{g \in C} \Gamma_g \backslash X^+ ,$$

where $X^+ \subseteq X$ is a connected component & $C \subseteq G(\mathbb{A}_f)$ is some finite set.

These nice assumptions & the exact definitions of both C & Γ_g can be described concretely, but I shall not do it here because of time-constraints.

§6. Hecke correspondence

For any $g \in G(\mathbb{A}_f)$ & $K \subseteq G(\mathbb{A}_f)$ compact open, we denote by $K' = K \cap gKg^{-1}$, we a natural correspondence:

$$Sh_{K'} \xrightarrow{(c_g,\, c_1)} Sh_K \times Sh_K$$

which induce a map on the cohomology groups of Sh_K. And so, we have an action

$$G(\mathbb{A}_f) \circlearrowleft H^i_{(c)}(Sh_K, *)$$

where $* = \mathbb{C}, \mathbb{Q}_p, \mathbb{Z}/p^N\mathbb{Z}$.

§7. Types

7.1 Hodge type:

Definition 13: A Shimura datum (G, X) is Hodge type if there is an injective homomorphism $G \hookrightarrow GSp_{2g}$ carrying X to $\mathcal{H}_g^{\pm}$.

The corresponding Shimura variety (also called Hodge type) of level $K \subseteq G(\mathbb{A}_f)$. are moduli spaces of families of the form

$$\left\{ (A, (s_i)_{0 \le i \le n}, \eta K) \right\} / \cong$$

- $A/\mathbb{C}$ is a complex abelian variety.
- s_0 or $-s_0$ is a polarization for $H_1(A, \mathbb{Q})$.
- $(s_i)_{1 \le i \le n}$ are morphisms of the form

$$H_1(A, \mathbb{Q})^{\otimes 2r} \longrightarrow \Lambda^{2r} H_1(A, \mathbb{Q}) \longrightarrow \mathbb{Q}(1).$$

- ηK is an orbit of an $\mathbb{A}_f$-linear iso $\mathbb{A}_f^{\oplus 2g} \longrightarrow TA \otimes \mathbb{Q}$ sending the multilinear form (s_i) to $\mathbb{A}_f^\times$-multiple of certain specific multilinear forms which determine $G \subseteq GSp_{2g}$.

These are the ones we will care about!

There is a subclass of these called the

PEL type Shimura varieties

- P → Polarization
- E → Endomorphism
- L → Level structure

$$\left\{\begin{matrix}\text{Siegel modular}\\ \text{varieties}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{PEL type}\\ \text{S.V.}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{Hodge type}\\ \text{S.V.}\end{matrix}\right\}.$$

Eg.: unitary S.V., geometric Hilbert modular varieties

Heutristically, these classiffy principally polarized abelian varieties (i.e. lattices in $\mathbb{C}^g$) with a predetermind "rationalised" Endomorphism ring + level structure.

We have seen that Shimura varieties are quotients of moduli spaces of Hodge structures. Conjecturally, all these Hodge-structure come from some kind cohomology, i.e. they come from motives (like in Hodge type). We do not know this in general, but we do know this for a larger class called abelian type.

$$\left\{\begin{matrix}\text{Siegel modular}\\ \text{varieties}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{PEL type}\\ \text{S.V.}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{Hodge type}\\ \text{S.V}\end{matrix}\right\} \subseteq \left\{\begin{matrix}\text{abelian}\\ \text{type S.U.}\end{matrix}\right\}$$

Eg: arithmetic Hilbert modular variety (I believe)

These are suppose to classify abelian motives with some principal polarization + level structure

§ 8. Automorphic line bundles

Since these varieties are defined over $\mathbb{C}$, there is no interesting Galois respresentation. Hence, we would like to construct "canonical" models over number fields

Their existence was shown by:

$\{\text{Hodg type}\} \subseteq \{\text{abelian type}\} \subseteq \{\text{general type}\}$

Deligne '71 — Deligne '79 — indep. by: Milne '83 & Borovoi '82/'83/'86 (not sure)

"the" number field mentioned here is called the reflex field (dual field originally)

- The Hecke action makes sense over the number field
- "special points" which correspond to CM abelian varieties/motives are already over the number field
- The Galois action on these special points come from the inverse of the reciprocity morphism of global CFT.

By Theorem 7 (Bailey-Borel), The Shimura varieties X_K have a minimal compactification X_K^*. By a theorem of Faltings, these can be defined over $\overline{\mathbb{Q}}$. One can define a natural ample line bundle ω_K on X_K^*, such that

When X_K is the moduli of abelian varieties with additional structure, then $\omega_K|_{X_K}$ is the determinant is the sheaf of invariant differential forms of degree one

$$H^0(X_K^*, \omega_K)$$

are complex autorphic forms.

For instance if we consider the group GSp_{2g}, & the flag variety Fl of all totally isotropic subspaces of $\mathbb{Q}^2$ w.r.t. the inner form $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$,

then, there is a map.

$$\begin{tikzcd} X_K \arrow[r] \arrow[d, dashed] & Fl \\ X_K^* \arrow[ur, dashed, "\pi_{HT,K}"'] & \end{tikzcd}$$

which captures the Hodge-Tate filteration on the abelian varieties & extends to the boundary.

Fl has a canonical ample line bundle ω,

& $\omega_K \cong \pi_{HT,K}^* \omega_K$.

Emerton & Calegari had a more general aproach to describing (p-adic) automorphic forms using a result by Franke which says that they can be viewed as Hecke Eigenvalues in

$$H^i(X_K, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p,$$

which shall be used in this seminar.