ALGEBRAIC STACKS

In classical algebraic geometry, we have the following definition of algebraic stacks.

<u>Definition</u>: A functor

$$X : \mathrm{Ring}^{\text{static}} \longrightarrow \mathrm{Gpd}$$

is an (fppf-) algebraic stack if it satisfies:

① X satisfies fppf-descent, i.e., for any fppf cover $\operatorname{Spec} B \xrightarrow{\varphi} \operatorname{Spec} A$, and any pair (E, σ) where $E \in X(B)$ and

$$\sigma : pr_1^* E \xrightarrow{\sim} pr_2^* E$$

over $B \underset{A}{\otimes} B$, such that over $B \underset{A}{\otimes} B \underset{A}{\otimes} B$

the following diagram commutes

$$\begin{tikzcd}
pr_{12}^* pr_1^* E \arrow[r, "pr_{12}^*\sigma"] \arrow[d, equal] & pr_{12}^* pr_2^* E \arrow[r, equal] & pr_{23}^* pr_1^* E \arrow[d, "pr_{23}^*\sigma"] \\
pr_{13}^* pr_1^* E \arrow[r, "pr_{13}^*\sigma"] & pr_{13}^* pr_2^* E \arrow[r, equal] & pr_{23}^* pr_2^* E
\end{tikzcd},$$

we can define a category $X(\operatorname{Spec} B \to \operatorname{Spec} A)$ of such pairs. Then, the functor

$$X(\operatorname{Spec} A) \xrightarrow{\varphi^*} X(\operatorname{Spec} B \to \operatorname{Spec} A)$$

is an equivalence.

② The diagonal map

$$\Delta : X \longrightarrow X \times X$$

is representable, i.e., for any $A \in \mathrm{Ring}^{\mathrm{Static}}$, the fiber product

$$\begin{tikzcd}
X \arrow[r] & X \times X \\
\mathfrak{X} \arrow[u, dashed] \arrow[r, dashed] \arrow[ur, phantom, "\lrcorner"] & \operatorname{Spec} A \arrow[u]
\end{tikzcd}$$

is representable by an algebraic space.

3. There is a smooth epimorphism $S \to X$, for some scheme S.

We wish define something similar in the derived setting.

<u>Notation:</u>

$\mathrm{Aff} := \mathrm{Ring}^{op}$ category of affine schemes.

And, for $R \in \mathrm{Ring}$, we denote by $\mathrm{Spec}\, R \in \mathrm{Aff}$ the corresponding object, and denote the underlying topological space by $|\mathrm{Spec}\, R|$.

<u>Definition:</u> The category of algebraic prestacks is the full subcategory

See Remark 5.2 in Camargo's notes for more details

$$\mathrm{PreStk} \subseteq \mathrm{Fun}(\mathrm{Ring}, \mathrm{Ani})$$

generated under (small) colimits by Aff via the Yoneda Embedding

Suppose $T \in \{\text{Zariski}, \text{étale}, \text{fppf}, \text{fpqc}\}$ is a topology on Aff (more generally, T is an "accessible" Grothendieck topology).

Moreover, for an uncountable regular caridinal $\kappa \geq \omega_1$, let Aff^{κ} be the category of κ-light affine schemes.

<u>Definition</u>: We define the category of κ-light algebraic stacks to be

$$\mathrm{AlgStk}_T^{\kappa\text{-light}} \subseteq \mathrm{Fun}(\mathrm{Aff}^{\kappa,\mathrm{op}}, \mathrm{Ani})$$

such that for any $X \in \mathrm{Aff}^{\kappa}$, and covering sieve $\mathcal{C}^{(0)}_{/X} \in \mathrm{Sieve}_T(X)$

$$F(X) \longrightarrow \lim_{Y \in \mathcal{C}^{(0),\mathrm{op}}_{/X}} F(Y)$$

is an equivalence.

Now, if $\kappa' \geq \kappa$ is another regular cardinal, then the left Kan extension

$$\mathrm{Fun}(\mathrm{Aff}^{\kappa,op}, \mathrm{Ani}) \subseteq \mathrm{Fun}(\mathrm{Aff}^{\kappa',op}, \mathrm{Ani}).$$

$$\begin{tikzcd}
 & \mathrm{Aff}^{\kappa',op} \arrow[dr, dashed, "T_{\kappa\to\kappa'}(F)"] \arrow[d, phantom, "\Uparrow\eta"] & \\
\mathrm{Aff}^{\kappa,op} \arrow[ur] \arrow[rr, "F"'] & {} & \mathrm{Ani}
\end{tikzcd}$$

defines $\mathrm{AlgStk}_T^{\kappa\text{-light}} \xrightarrow{T_{\kappa\to\kappa'}} \mathrm{AlgStk}_T^{\kappa'\text{-light}}$, such that $T_{\kappa\to\kappa'}$ is a fully-faithful, left exact colimit preserving functor.

Then, we define the category of algebraic stacks as

$$\mathrm{AlgStk}_T := \operatorname*{colim}_{\kappa} \mathrm{AlgStk}_T^{\kappa\text{-light}}$$

along $T_{\kappa\to\kappa'}$ for all uncountable regular cardinals κ

Example : Classical algebraic stacks.

Let $\mathrm{Aff}_{\leq n} \subseteq \mathrm{Aff}$ be the full subcategory of n-truncated affine schemes, and $T \in \{\text{Zariski, étale, fppf, fpqc}\}$ as before.

Let $\mathrm{Sh}_T(\mathrm{Aff}_{\leq n}, \mathrm{Ani})$ be the category of sheaves on $\mathrm{Aff}_{\leq n}$ (defined in a manner similar to AlgSt_T using regular cardinals).

Then, we have a fully faithful embedding of sites $\mathrm{Aff}_{\leq n, T} \subseteq \mathrm{Aff}$ which induces a left exact, colimit preserving functor

$$F_n : \mathrm{Sh}_T(\mathrm{Aff}_{\leq n}, \mathrm{Ani}) \longrightarrow \mathrm{AlgStk}_T .$$

An object $X \in \mathrm{AlgStk}_T$ is called n-classical if it is in the essential image of F_n.

Further, it is called classical if it is 0-classical.

<u>Lemma</u>: Let $X = \mathrm{Spec}\, R$ be an affine scheme viewed as an algebraic prestack $X \in \mathrm{PreStk}$ via the Yoneda embedding. The X is an fpqc-stack.

<u>Slogan</u>: The fpqc topology is subcanonical

Consider the Zariski spectrum functor defined previously,

$$|\cdot| : \mathrm{Aff} \longrightarrow \mathrm{Top}.$$

Then, (because our categories are nice) we can define a left Kan extension

$$\begin{tikzcd}
 & \mathrm{AlgStk}_{fpqc} \arrow[dr, dashed] & \\
\mathrm{Aff} \arrow[ur] \arrow[rr] \arrow[rr, phantom, shift left=4ex, "\Uparrow"] & & \mathrm{Top}
\end{tikzcd}$$

And, we continue to denote and call the functor

$$|\cdot| : \mathrm{AlgStk}_{fpqc} \longrightarrow \mathrm{Top}$$

the underlying Zariski spectrum functor.

Now, suppose $T \in \{\text{Zariski}, \text{étale}, \text{fppf}, \text{fpqc}\}$ and $Y \in \mathrm{AlgStk}_T$. If $|Y|$ is the underlying Zariski spectrum and $|U| \subseteq |Y|$ is an open subspace, we define the the substack, whose $\mathrm{Spec}\, R$ points are

$$\mathcal{U}(R) = \begin{cases} Y(R) & \text{if } |\operatorname{Spec} R| \to |Y| \text{ factors through } \mathcal{U} \\ \emptyset & \text{otherwise.} \end{cases}$$

Definition: A scheme X is a Zariski algebraic stack, i.e., $X \in \mathrm{AlgStk}_{\mathrm{Zar}}$, such that $|X|$ admits an open cover $\{|\mathcal{U}_i|\}_i$ such that $\mathcal{U}_i$'s are affine schemes. We shall denote by $\mathrm{Sch} \subseteq \mathrm{AlgStk}_{\mathrm{Zar}}$ the full subcategory of schemes.

Proposition: Let $X \in \mathrm{Sch}$ be a scheme. Then X is a sheaf for the fpqc topology. In particular, we have a fully faithful embedding $\mathrm{Sch} \subseteq \mathrm{AlgStk}_{\mathrm{fpqc}}$.

Consider the functor of derived catgories of modules.

$$D : \mathrm{Aff}^{op} \longrightarrow \mathrm{CAlg}(\mathrm{Cat}),$$

where $D(X)$ is a symmetric monoidal category for $X \in \mathrm{Aff}$

<u>Definition:</u> We define the functor of quasicoherent sheaves on AlgStk_{fpqc} to be the right Kan extension of D

$$\begin{tikzcd}
 & \mathrm{AlgStk}_{fpqc}^{op} \arrow[dr, dashed, "D"] \arrow[d, phantom, "\Downarrow \varepsilon"] & \\
\mathrm{Aff}^{op} \arrow[ur] \arrow[rr] & {} & \mathrm{CAlg}(\mathrm{Cat})
\end{tikzcd}$$

Given a morphism $f : Y \to X$ of algebraic stack, we let $f^* = D(f) : D(X) \longrightarrow D(Y)$

be the pullback functor (which is symmetric monoidal), and let $f_* : D(Y) \to D(X)$ be its right adjoint (the relative cohomology functor).

Further, we denote by $\otimes_X$ the tensor product of $D(X)$.

Futher, we let $\underline{\mathrm{Hom}}(\cdot,\cdot)$ be the internal Hom functor of $D(X)$ such that

$$- \otimes_X M \dashv \underline{\mathrm{Hom}}(M, -).$$

<u>Proposition</u>: The functor $D : \mathrm{Aff}^{op} \to \mathrm{CAlg}(\mathrm{Cat})$ is a sheaf for the fpqc topology. In particular its right Kan extension

$$D : \mathrm{AlgStk}^{op} \to \mathrm{CAlg}(\mathrm{Cat})$$

preserves limits.

Let $X \in \mathrm{AlgStk}_{fpqc}$.

We define $D_{\geq 0}(X) \subseteq D(X)$ the full subcategory of those quasi-coherent sheaves $M \in D(X)$ such that for any map $f : \operatorname{Spec} R \to X$, $f^* M \in D_{\geq 0}(R)$.

Now, let $A \in \mathrm{Ring}$. Recall, that we have an adjunction

$$\mathrm{Sym}^{\bullet}_A : D_{\geq 0}(A) \rightleftarrows \mathrm{Ring}_{A/} : G(A)$$

<u>Lemma</u>: The functor

$$\mathrm{Ring}_{(-)/} : \mathrm{Aff}^{op} \longrightarrow \mathrm{Cat}$$

sending $\mathrm{Spec}\, R$ to $\mathrm{Ring}_{R/}$ satisfies fpqc descent. Moreover, the natural transformation

$$G(-) : \mathrm{Ring}_{(-)/} \longrightarrow D_{\geq 0}(-)$$

has a left adjoint given by the symmetric powers functor.

$$\mathrm{Sym}^{\bullet}_{(-)} : D_{\geq}(-) \longrightarrow \mathrm{Ring}_{(-)/}.$$

<u>Definition</u>: The functor of relative animated algebras over fpqc algebraic stacks is defined to be the right Kan extension of the $\mathrm{Ring}_{(-)/}$ functor.

$$\begin{tikzcd} & \mathrm{AlgStk}_{fpqc}^{op} \arrow[dr, dashed, "\mathrm{Ring}_{(-)}"] \arrow[d, phantom, "\Downarrow \varepsilon"] & \\ \mathrm{Aff}^{op} \arrow[ur] \arrow[rr] & {} & \mathrm{Cat} \end{tikzcd}$$

The forgetful natural transformation $G(-) : \mathrm{Ring}_{(-)} \longrightarrow D_{\geq 0}(-)$ of functors from $\mathrm{AffStk}_{fpqc}^{op} \longrightarrow \mathrm{Cat}$ is obtained as the right Kan extension of $G(-)$ for Affine schemes.

Moreover, it has a left adjoint given by the right Kan extension of the symmetric power functor

$$\mathrm{Sym}^{\bullet}_{(-)} : D_{\geq 0}(-) \longrightarrow \mathrm{Ring}_{(-)}$$

<u>Definition</u>: Suppose $X \in \mathrm{AlgStk}_{fpqc}$ and $\mathcal{A} \in \mathrm{Ring}_X$. Then, the relative spectrum of $\mathcal{A}$ over X, denoted by $\mathrm{Spec}_X \mathcal{A}$ is the fpqc stack in AlgStk_{fpqc}/X whose values at $f : \mathrm{Spec}\, R \longrightarrow X$ are given by.

$$(\mathrm{Spec}_X \mathcal{A})(R) = \mathrm{Map}_{\mathrm{Ring}_R}(f^* \mathcal{A}, R)$$