ANALYTICS RINGS AND ANALYTIC STACKS

Ref: Analytic stacks lectures : 8, 9, 15, 16, 17, 18, 19.

§1. Recollection

Notation :

$$P = \mathbb{Z}[\mathbb{N} \cup \{\infty\}] / \mathbb{Z}[\infty].$$ as a condensed abelian group.

For a condensed abelian group (set might suffice). M, $\mathrm{Hom}(P, M)$ detects null sequences.

Remark : If M is a non-archimedean complete. topological abelian group, then the map

$$(\mathrm{id} - \mathrm{shift}) : \{\text{null- sequences}\} \longrightarrow \{\text{null sequences}\}$$

$$(m_1, m_2, \dots) \longmapsto (m_1 - m_2, m_2 - m_3, \dots)$$

is invertible (because their sum's converge)

Definition: A object M in CAb (condensed ab. groups) is solid if:

$$(\mathrm{id}_P - \sigma)^* : \mathrm{Hom}_{CAb}(P, M) \longrightarrow \mathrm{Hom}_{CAb}(P, M)$$

is an isomorphism (σ = shift).

i.e. if (a_n) is a null sequence in M, then $\sum a_n$ converges.

We shall denote this by $\mathrm{Solid}_{\mathbb{Z}}$.

Now to Huber's theory from previous semesters.

A Huber ring is a topological ring $A^{\triangleright}$ with the following data:

$$\begin{tikzcd}
 & A^{\triangleright} & \\
A_0 \arrow[ur, phantom, sloped, "\subseteq"] \arrow[rr, phantom, "\subseteq"] & & A^{\circ} \arrow[ul, phantom, sloped, "\subseteq"] \\
I \arrow[u, phantom, sloped, "\subseteq"] \arrow[rr, phantom, "\subseteq"] & & A^{\circ\circ} \arrow[u, phantom, sloped, "\subseteq"]
\end{tikzcd}$$

Ring of definition A_0; A° Ring of power bounded elements

Ideal of definition I; $A^{\circ\circ}$ ideal of topologically nilpotent elements.

Huber pair : $A = (A^\triangleright, A^+)$ ring of integral elements.

These were the building blocks of adic spaces.

We would like to build a more generalised theory using condensed machinery.

§2. Analytic rings

Definition: A <u>condensed (commutative) ring</u> is a ring object $R^\triangleright$ in the symmetric monoidal category CAb, and a <u>condensed $R^\triangleright$-module</u> is an $R^\triangleright$-module object in CAb. We shall denote by $\mathrm{Mod}_{R^\triangleright}$, the subcategory of CAb of such objects.

Definition: An <u>analytic ring</u> is a pair $R = (R^\triangleright, \mathrm{Mod}_R)$, where $R^\triangleright$ is a condensed commutative ring, and Mod_R is a full subcategory of $\mathrm{Mod}_{R^\triangleright}$, which is:

(i) closed under limits, colimits and extensions.

(ii) $\forall M \in \mathrm{Mod}_{R^\triangleright}$ and $N \in \mathrm{Mod}_R$, $\mathrm{Ext}^i_{R^\triangleright}(M, N) \in \mathrm{Mod}_R$ for all $i \geq 0$.

(iii) $R^\triangleright \in \mathrm{Mod}_R$.

Think of $(R^\triangleright, \mathrm{Mod}_R)$ as the condensed analogue of $(R^\triangleright, R^+)$ in Huber's theory.

A <u>morphism of analytic rings</u> $(R^\triangleright, \mathrm{Mod}_R) \to (S^\triangleright, \mathrm{Mod}_S)$ is a ring morphism $R^\triangleright \longrightarrow S^\triangleright$ such that the composition $\mathrm{Mod}_S \hookrightarrow \mathrm{Mod}_{S^\triangleright} \longrightarrow \mathrm{Mod}_{R^\triangleright}$ has essential image in Mod_R.

Additionally, an analytic ring is called solid, if objects in $\mathrm{Mod}_R \subseteq \mathrm{CAb}$ are so.

Examples:

- $(\mathbb{Z}, \ \mathrm{Solid}_{\mathbb{Z}})$

- For any condensed commutate ring R, we define the solid R-modules
$$\mathrm{Mod}_{R\square} := \mathrm{Mod}_R(\mathrm{Soild}_{\mathbb{Z}}).$$
Then $(R, \mathrm{Mod}_{R\square})$ is an analytic ring.
When R corresponds to a discrete ring, these will correspond to affine schemes.

- $(\mathbb{R}, \mathrm{Liquid}_p)$ & I believe we have a similar result over $\mathbb{C}$. $(0 < p \leq 1)$.

- "Huber pairs" (might need some analytic/Tate conditions)

Proposition: $(A^{\triangleright}, \mathrm{Mod}_A)$ – analytic ring, then $\exists$ left adjoint

$$\begin{aligned} \mathrm{Mod}_{A^{\triangleright}} &\longrightarrow \mathrm{Mod}_A \\ M &\longmapsto M \otimes_{A^{\triangleright}} A \end{aligned}$$

to the inclusion. The kernel is a $\otimes$-ideal, and Mod_A aquires a unique symmetric monoidal structure making $- \otimes_{A^{\triangleright}} A$ symmetric monoidal.

<u>Proof idea</u>: Existence is just abstract nonsense.

Let $M \in \mathrm{Mod}_{A^{\triangleright}}$ s.t. $M \otimes_{A^{\triangleright}} A = 0$.

Need to show: For all $N \in \mathrm{Mod}_{A^{\triangleright}}$, $(M \otimes_{A^{\triangleright}} N) \otimes_{A^{\triangleright}} A = 0$

$\iff \forall L \in \mathrm{Mod}_A$, $\mathrm{Hom}_{A^{\triangleright}}(N \otimes_{A^{\triangleright}} M, L) = 0$. L is techically inclusion(L)

$$\mathrm{Hom}_{A^{\triangleright}}(N \otimes_{A^{\triangleright}} M, L) = \mathrm{Hom}_{A^{\triangleright}}(M, \underbrace{\underline{\mathrm{Hom}}_{A^{\triangleright}}(N, L)}_{\in \mathrm{Mod}_A})$$

$$\overset{\text{just adjunction}}{=} \mathrm{Hom}_{\mathrm{Mod}_A}(M \otimes_{A^{\triangleright}} A, \ldots\ldots) = 0$$

So, $\forall\, M, N \in \mathrm{Mod}_A$.

$$M \underset{A}{\otimes} N := \left(M \underset{A^\triangleright}{\otimes} N \right) \underset{A^\triangleright}{\otimes} A.$$

Need to check that it is symmetric monoidal ....

$\boxtimes$

The derived category.

Let A be an analytic ring. Then,

$$D(A) \subseteq D(\mathrm{Mod}_{A^\triangleright})$$

be the fullsubcat of all $M \in D(\mathrm{Mod}_{A^\triangleright})$ s.t. $H_i(M) \in \mathrm{Mod}_A$.

Properties: ① $\exists$ functor $D(\mathrm{Mod}_A) \longrightarrow D(A)$, but it not always an equivalence.

② $D(A)$ is triangulated (stable ∞-cat once we switch to that language) stable under all $\bigoplus$ and $\prod$.

3. The inclusion has a left adjoint

$$- \overset{L}{\underset{A^{\blacktriangleright}}{\otimes}} A : D(\mathrm{Mod}_{A^{\blacktriangleright}}) \longrightarrow D(A).$$

where the kernel is a $\otimes$-ideal.

$$\begin{tikzcd} & \text{Solid rings} \arrow[dr] & \\ \text{Huber rings} \arrow[ur] \arrow[rr, hook] & & \text{Condensed rings} \end{tikzcd}$$

Let $A^{\blacktriangleright}$ be a solid ring. Can define

$$A^{\circ\circ} \subseteq A^{\circ} \subseteq A^{\blacktriangleright}(*).$$

$$A^{\circ\circ} = \left\{ f \in A^{\blacktriangleright}(*) \;\middle|\; \begin{tikzcd} \mathbb{Z}[T] \arrow[r, "T \mapsto f"] \arrow[d] & A^{\blacktriangleright} \\ \mathbb{Z}[[T]] \arrow[ur, dashed, "\exists"'] & \end{tikzcd} \right\}$$

$$= \left\{ f \in A^{\blacktriangleright}(*) \;\middle|\; \sum a_n f^n \text{ converges} \right\}.$$

$$A^{\circ} = \left\{ f \in A^{\blacktriangleright}(*) \;\middle|\; \mathbb{Z}[T] \xrightarrow{T \mapsto f} A^{\blacktriangleright} \text{ makes } A^{\blacktriangleright} \text{ a } \mathbb{Z}[T]\text{-solid module} \right\}.$$

$$= \left\{ f \in A^\triangleright(*) \;\middle|\; \begin{array}{l} \mathbb{Z}[T] \to A^\triangleright \text{ is such that} \\ (1-\sigma)^* \text{ and } (1-f\sigma)^* \text{ gives an} \\ \text{iso of } \mathrm{Hom}(P \otimes R, A^\triangleright) \end{array} \right\}$$

$$= \left\{ f \in A^\triangleright(*) \;\middle|\; \begin{array}{l} \text{For all null-sequences } (m_n). \\ \sum m_n \text{ and } \underbrace{\sum f^n m_n}_{\Downarrow\ f \text{ is bounded}} \text{ converge} \end{array} \right\}$$

When $A^\triangleright$ comes from a Huber ring, we recover the classical notions.

Given an solid analytic ring $A = (A^\triangleright, \mathrm{Mod}_A)$ for a Huber ring $A^\triangleright$, we can also define $A^+ \subseteq A^\triangleright(*)$.

Proprosition: There is a fully faithful embedding

$$\{(\text{Tate}) \text{ Huber pairs}\} \longleftrightarrow \{\text{solid analytic ring}\}.$$

$$(A^\triangleright, A^+) \longmapsto (A^\triangleright, A^+)_\blacksquare$$

which has a right adjoint (captured above).

§3. Analytic stacks

Talk 2

Let AnRings be the category of Analytic rings (they switch to animated rings, but we will not).

<u>Want</u>: A topology on

$$\mathrm{AffAn} := \mathrm{AnRings}^{op}$$

such that the following are (∞-) sheaves

① schemes.

② complex analytic spaces.

③ Adic spaces / Banach rings

⋮

And,

$$X \longmapsto D(X)$$

satisfies a six-functor formalism.

Simply put, we want:

(i) $\forall X \in \mathrm{AffAn}$, $D(X)$ has a symmetric $\otimes$ and $\underline{\mathrm{Hom}}$ such that

$$\otimes \dashv \underline{\mathrm{Hom}}$$

(ii) $\forall f : X \to Y$ is AffAn, there is a $f^* : D(Y) \to D(X)$ and $f_* : D(X) \to D(Y)$ such that

$$f^* \dashv f_*$$

(iii) There is a class of morphisms called !-able $f : X \to Y$ which have $f^! : D(Y) \to D(X)$ and $f_! : D(X) \to D(Y)$, such that

$$f_! \dashv f^!$$

Such that

(a) Base change:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd} \qquad g^* f_! \cong g'^* f'_!$$

(6) Projection formula: $\forall A \in D(X)$ & $B \in D(Y)$.

and nice (!-able) $f : X \longrightarrow Y$.

$$f_!(A \otimes f^* B) \cong f_! A \otimes B.$$

!-able morphism in the classical setting

(i) we have a class of proper morphisms where we define $f_! = f_*$

(ii) we have a class of open morphisms where $f^!$ is the left adjoint of f^*.

(iii) A general map $X \xrightarrow{f} Y$ is !-able if it factors through Y':

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "j"'] & & Y \\ & \bar{Y} \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

where j is open;

and $\bar{f}$ is proper.

<u>Theorem</u> (Liu-Zhang, Mann): Under minimal assumptions on the class of "proper maps", "open immersions" and "!-able maps", determines an abstract 6-functor formalism.

Notation: For $A \in \mathrm{AnRings}$, we denote the corresponding object in $\mathrm{AnRings}^{op}$ by $\mathrm{AnSpec}\, A$.

Definition: A morphism $f: \mathrm{AnSpec}\, B \to \mathrm{AnSpec}\, A$ is called proper if $f_*: D(B) \longrightarrow D(A)$ satisfies the projection formula.

Definition: A map $j: \mathrm{AnSpec}\, B \to \mathrm{AnSpec}\, A$ is a open immersion if j^* admits a fully-faithful left adjoint $j_!$ satisfying the proj. form.

Definition: A map $f: X \to Y$ is $!$-able if it factors through an $\overline{X}$

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "j"'] & & Y \\ & \overline{X} \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

where j is an open immersion, and $\overline{f}$ is proper.

Remark: The above notion of $!$-able maps will give us a 6-functor formalism

$$\operatorname{AnSpec} A = X \longmapsto D(X) = D(A).$$

Definition: A !-able map $f: X \to Y$ is said to be a !-cover if it satisfies !-descent, i.e.

$$D(Y) \xrightarrow[f^!]{\sim} \underbrace{\underset{\overleftarrow{\Delta}}{\lim} \left(D(X) \underset{p_2^!}{\overset{p_1^!}{\rightrightarrows}} D(X \times X) \,\substack{\rightarrow\\ \rightarrow\\ \rightarrow}\, \cdots \right)}_{\overset{?}{=} \ \underset{\Delta}{\lim^!} D(X)}$$

Topology

A set $\{X_i \to X\}_{i \in I}$ is said to be a cover, if there exists a finite subset $I_0 \subseteq I$ such that $f_i : X_i \to X$ is !-able for all $i \in I_0$, and

| can use sieves

$$\coprod_{i \in I_0} X_i \longrightarrow X.$$

is a !-able cover.

All cases that we are interested in are sheaves on AnRings with the !-topology given previous. But, analytic stacks are defined a bit more generally using the language of animated rings, anima and ∞-categories.

Definition: An analytic stack is an accessible functor

F is said to be accessible if $\exists$ a cardinal κ, s.t. F commutes w/ κ-filtered colimits

$$X : \text{AnRings} \longrightarrow \underset{(\text{Anima})}{\text{Sets}}$$

such that for all (hyper) covers

$$\text{AnSpec}(A_\bullet) \longrightarrow \text{AnSpec}(A),$$

for which

$$D(A) \xrightarrow{\sim} \lim{}^{!} D(A_\bullet),$$

we have

$$X(A) \xrightarrow{\sim} \lim X(A_\bullet)$$

Analytic spaces are analytic stack which can locally be seen as isomorphic to the functor $h_{(A^\triangleright, \mathrm{Mod}_A)}$.

Schemes, manifolds and (analytic) adic spaces are examples of analytic spaces.