Moduli of elliptic curves

What is a moduli space?

It is a "space" whose points represent objects you want to classify.

Eg: Lattices in $\mathbb{C}$.

Definition: A (rank 2) Lattice in in $\mathbb{C}$, Is a free $\mathbb{Z}$-module in $\mathbb{C}$ of rank 2, which generates $\mathbb{C}$ as an $\mathbb{R}$-vector space.

We say 2 lattices L_1 & L_2 are homothetic if there exists a $\lambda \in \mathbb{C}^\times$ such that

$$L_2 = \lambda L_1$$

Pick a lattice L, and an ordered basis (ω_1, ω_2). We can then define a map

$$\mathbb{C} \setminus \mathbb{R} =: \mathcal{H}^{\pm} \twoheadrightarrow \{\text{Lattices}\}/_{\sim}$$

$$\omega \longmapsto [\mathbb{Z}\omega + \mathbb{Z}]$$

where $$\omega_1/\omega_2 \longmapsto [L]$$

Consider another ordered basis (ω_1', ω_2') of L. Then, the exists a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}.$$

Hence, $$\omega_1'/\omega_2' = \frac{a(\omega_1/\omega_2) + b}{c(\omega_1/\omega_2) + d}.$$

Inspired by this, we define the möbius transformation, i.e., an action $GL_2(\mathbb{Z}) \circlearrowright \mathcal{H}^{\pm}$, where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az+b}{cz+d}$$

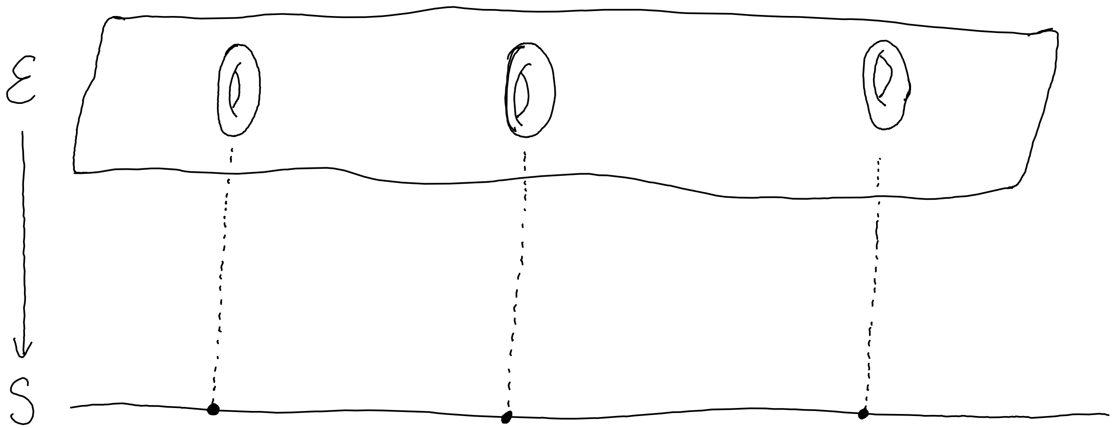
for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$, and $z \in \mathcal{H}^{\pm}$. And, we get a moduli spaces of lattices in $\mathbb{C}$.

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{Lattice in} \\ \mathbb{C} \end{array} \right\}\Big/\sim & \overset{\cong}{\longleftrightarrow} & GL_2(\mathbb{Z}) \backslash \mathcal{H}^{\pm} \\ \updownarrow \cong & & \\ \left\{ \begin{array}{l} \text{the quotient} \\ \text{groups } \mathbb{C}/\Lambda, \\ \text{for } \Lambda \subseteq \mathbb{C} \text{ a lattice} \end{array} \right\} & \cong & \{\text{elliptic curves}/\mathbb{C}\} \end{array}$$

So, elliptic curves over $\mathbb{C}$ are parametrized by morphisms/functions

$$\{*\} \longrightarrow GL_2(\mathbb{Z}) \backslash \mathcal{H}^{\pm}$$

We can infact define relative elliptive curves, that look like:



Then, a moduli space of elliptic curves is a functor

$$\begin{aligned} \mathcal{M} : Sch^{op} &\longrightarrow Sets \\ S &\longmapsto \{\mathcal{E}/S\}/_{\cong} \end{aligned}$$

which, for some "good" topology on Sch, satisfies the sheaf condition.

<u>Definition</u>: An elliptic curve over a scheme S, is a proper, smooth map

$$\mathcal{E} \longrightarrow S,$$

along with "point" $S \xrightarrow{e} \mathcal{E}$, such that

$$\begin{tikzcd} S \arrow[r, "e"] \arrow[dr, "id"'] & \mathcal{E} \arrow[d] \\ & S \end{tikzcd}$$

the fiber over a point (i.e., the pre-image of a point) is a connected genus one curve.

<u>Note</u>: $\mathcal{E}/S$ is an abelian group object in Sch/S.

There is a very nice topology on the category Schemes, called the étale topology. Amongst many, there are two reasons why it is nice:

1. If K'/K is a finite separable extension, then
$$X_{K'} \longrightarrow X_K$$
is a cover. This will be important soon.

2. $$\mathrm{Hom}(\cdot, X) : \mathrm{Sch}_{\acute{e}t}^{op} \longrightarrow \mathrm{Set}$$
is a étale sheaf.

Unfortunately, $\mathcal{M}$ as defined before is not a sheaf on the étale site.

Let's see why. There exists non-isomorphic elliptic curves, say E_1, E_2 over $\mathbb{Q}$, which are isomorphic over a finite separable extension $K/\mathbb{Q}$.

So, The map,

$$\mathcal{M}(\mathbb{Q}) \longrightarrow \mathcal{M}(K)$$

is not injective, which it would be if it were a sheaf, because then

$$\mathcal{M}(\mathbb{Q}) = \lim \left(\mathcal{M}(K) \rightrightarrows \mathcal{M}(K \otimes_{\mathbb{Q}} K) \right).$$

We will see a geometric interpretation of this later.

<u>A fix to this problem</u>: Keep track of the isomorphisms.

So, from now on:

$$\mathcal{M} : \mathrm{Sch}^{op} \longrightarrow \mathrm{Gpd}$$

$$S \longmapsto \left\{ \begin{array}{l} \text{ob} : \mathcal{E}/S \ \text{elliptic curves} \\ \text{mor} : \text{Isomorphisms over } S \end{array} \right\}$$

which is a stack (2-sheaf). It is in fact a smooth separated Deligne-Mumford stack over $\mathbb{Z}$.

But is this moduli a "space"?

we have answered this question already, but let us take a look at it again.

We have seen that

$$\{\text{Elliptic curves}/\mathbb{C}\}/\cong \xrightarrow{\cong} \mathrm{Hom}(*, GL_2(\mathbb{Z})\backslash\mathcal{H})$$

So, does there exist a spaces $X \in Sch$ such that for every field K

$$\mathcal{M}(K) := \mathcal{M}(\mathrm{Spec}\,K) \xrightarrow{\cong} \mathrm{Hom}(\mathrm{Spec}\,K, X) =: X(K)\,?$$

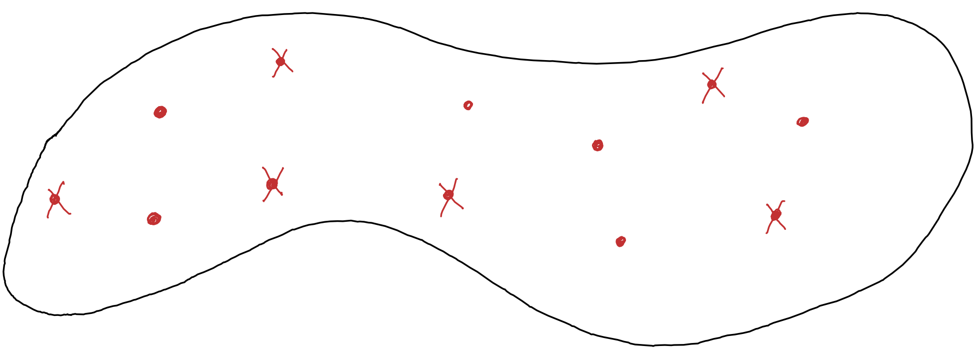
or, more generally, for any $S \in Sch$

$$\mathcal{M}(S) \xrightarrow{\cong} \mathrm{Hom}(S, X) =: X(S)\,?$$

Answer: No! Because the latter is a sheaf.

A Geometric interpretation of why we cannot get a "space".

Suppose $\mathcal{M}$ is determined by a scheme.



× — $M(\mathbb{Q})$ [= solutions to polynomials over $\mathbb{Q}$]

• — $M(K)$ [= solutions to polynomials over K]

$K/\mathbb{Q}$ a a finite separable extension

Since $\mathbb{Q} \subseteq K$, every solution over $\mathbb{Q}$ is also a solution over K. Hence $\mathcal{M}(\mathbb{Q}) \subseteq \mathcal{M}(K)$. But, this not true.

What we have just seen is that $\mathcal{M}$ is not representable by a scheme.

<u>Definition</u>: We say that a moduli problem $\mathcal{M}$ is representable by a scheme, if there exists a scheme X such that we have a natural isomorphism

$$\mathrm{Hom}(\cdot\,, X) \xrightarrow{\sim} \mathcal{M}(\cdot),$$

i.e., $\mathcal{M}$ is in the essential image of the Yoneda embedding

$$\mathrm{Sch} \hookrightarrow (\mathrm{P})\mathrm{Sh}(\mathrm{Sch}).$$

We shall now add aditional structures to make them representable. Consider the category $\mathcal{E}ll$, where:

Objects: $\begin{matrix} \mathcal{E} \\ \downarrow \\ S \end{matrix}$ elliptic curves over a base S.

Morphisms: $\begin{tikzcd} \mathcal{E}' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{E} \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$ Cartersian diagrams

A level structure is a functor

$$F : \mathcal{E}ll^{op} \longrightarrow Sets$$

A moduli problem of elliptic curves with level structure F is representable if there is a scheme X_F such that there is a natural isomorphism between $Hom(\cdot, X_F)$ and the functor:

$$\begin{aligned} Sch &\longrightarrow Gpd \\ S &\longmapsto \{ (\mathcal{E}/S, \lambda \in F(\mathcal{E}/S) \} \end{aligned}$$

Exercise: A moduli problem with level structure F is representable, only if it is rigid, i.e., for all $\mathcal{E}/S \in \mathcal{E}ll$ and $\alpha \in F(\mathcal{E}/S)$,

$$\mathrm{Aut}(\mathcal{E}/S, \alpha) = \{\mathrm{Id}\}$$

In this exercise, it is important that our moduli problem is valued in Groupoid and not sets (by taking isomorphism classes). This is because it is possible to find moduli problems which are not rigid, hence not representable, but if you consider isomophism classes, they suddenly become representable. This is because taken isomorphism classes is akin to just forgetting non-trivial automorphisms
Katz, Mazur — A.4.1.3

Suppose it is representable by X_F. Then, the following categories are equivalent

$$\underbrace{\mathrm{Hom}(S, X_F)}_{\text{the set as a category with only identity morphisms}} \xrightarrow{\sim} \underbrace{\{(\mathcal{E}/S, \lambda)\}}_{\text{the groupoid.}}$$

The equivalence implies that the only automorphisms on the right hand side are the identity maps

Examples of representable moduli problems:

1. The "naive" level structure

Let $N \geq 1$

$$F : \mathcal{E}/S \longmapsto \left\{ \begin{array}{c} S\text{-group scheme isos} \\ \underline{\mathbb{Z}/N\mathbb{Z}}_S \xrightarrow{\simeq} \mathcal{E}[N] \\ \text{where } \mathcal{E}[N] \text{ are the} \\ N\text{-torsion in } \mathcal{E} \end{array} \right\}$$

The moduli problem with the naive level structure is rep$^{\text{d}}$ by a smooth affine scheme in $\mathrm{Sch}/_{\mathbb{Z}[N^{-1}]}$ for $N \geq 3$.

$$X_F(\mathbb{C}) = \Gamma(N) \backslash \mathcal{H}^{\pm}$$

where $\Gamma(N) = \ker(GL_2(\mathbb{Z}) \to GL_2(\mathbb{Z}/N\mathbb{Z}))$

2. Weierstrass moduli.

Define:

$$F : \mathcal{E}/S \longmapsto \left\{ \begin{array}{c} \omega \in \Gamma(\mathcal{E}, \Omega^1_{\mathcal{E}/S}) \\ \text{generator} \end{array} \right\}$$

The moduli problem with the Weistrass level structure is representable by the affine scheme

$$\mathrm{Spec}\left(\mathbb{Z}[1/6][a,b][\Delta^{-1}]\right)$$

where $\Delta = 4a^3 + 27b^2$.

$$X_F(\mathbb{C}) = {}_{GL_2(\mathbb{Z})}\backslash \mathcal{H}^{\pm}$$

These are example of modular curves, which are special cases of space that (conjecturally) form moduli spaces of motives with level structure, called Shimura Varieties.

THE END