TALK 2

p-divisible & formal groups

§1. p-divisible groups

Fix prime p & base scheme S.

Definition: An fppf-abelian sheaf G over Sch/S is called p-divisible if for any $T \in Sch/S$ and any $x \in G(T)$, there is an fppf cover $T' \to T$ and $y \in G(T')$ such that $py = x|_{T'}$.

also called Barsotti-Tate

Definition: A p-divisible fppf-abelian sheaf G over Sch/S is called a p-divisible group if $G = \varinjlim_n G[p^n]$, where $G[p^n] := \ker(G \xrightarrow{[p^n]} G)$ are finite flat group schemes over S.

<u>Example</u>: ⊛ A/S an abelian variety is a p-divisible sheaf, but not a p-divisble group.

⊛ If A/S as before, then

$$A[p^\infty] := \varinjlim_n A[p^n].$$

If $S = \mathrm{Spec}(\overline{\mathbb{F}_p})$ & A is ordinary, then

$$A[p^\infty] = \hat{A} \times \left(T_p(A) \underset{\mathbb{Z}_p}{\times} \mathbb{Q}_p/\mathbb{Z}_p\right)$$

<u>Remarks</u>:

1. $\mathbb{Q}_p/\mathbb{Z}_p = \left(\mathbb{Z}/p\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^3\mathbb{Z} \to \cdots\cdots\right)$
2. But what is $\hat{A}$?

§2. Formal Lie groups

Let k be some field.

Let X/k be some nice smooth variety.

For $x \in X(k)$:

"Derivatives" of functions at x are calculated using

$$\operatorname{Spec} k[\varepsilon]/_{\varepsilon^2} \xrightarrow{x^\bullet} X$$

$\longrightarrow$ nilpotent elements in a ring capture fuzz.

$$X_x^\wedge = \operatorname{Spf} k[[t_1, \dots, t_d]] \left(\equiv \varinjlim \operatorname{Spec}\left(\mathcal{O}_X / I_x^n\right)\right) \to X.$$

Stack 05D9: (R ring).

X/R is a smooth scheme, $x \in X(R)$

$$\Rightarrow \quad X_x^\wedge = \operatorname{Spf} R[[t_1, \dots, t_d]]$$

Some constructions (R is ring).

I $G/_R$ is smooth group scheme (of dim d).

$e \in G(R)$ identity.

$$\hat{G} := G_e^{\wedge} = \mathrm{Spf}\, R[[t_1, \dots, t_d]]$$

The map $m : G \times G \longrightarrow G$ also "decends":

$$\begin{array}{ccc} \hat{m} : \hat{G} \times \hat{G} & \longrightarrow & \hat{G} \\ \| & & \| \\ \mathrm{Spf}\, \underbrace{R[[x_1, \dots, x_d]] \hat{\otimes} R[[y_1, \dots, y_d]]}_{\begin{array}{c}\cong \\ R[[x_1, \dots, x_d, y_1, \dots, y_d]]\end{array}} & \longrightarrow & \mathrm{Spf}\, R[[t_1, \dots, t_d]] \end{array}$$

$$\hat{m} \equiv \left\{ F_i(x_1, \dots, x_d, y_1, \dots, y_d) \in R[[x_1, \dots, x_d, y_1, \dots, y_d]] \right\}_{i=1}^{d}$$

II This family has certain properties :

Notation: $F = (F_1, \dots, F_d)$

$x = (x_1, \dots, x_d)$ (& y & z).

(i) $F(x, y) = x + y$ + higher deg terms

(ii) $F(x, F(y, z)) = F(F(x,y), z)$

Lets call such families d-dim'l FGLs.

An FGL is commutative if

(iii) $F(x,y) = F(y,z)$.

From here on, all groups are abelian

III Functorial interpretation of FGL

F d-dim'l FGL

$$F : \{R\text{-algebras}\} \longrightarrow Ab$$

$$S \longmapsto \left(\begin{array}{c} Nil(S)^d \\ \text{group struct given by } F \end{array} \right)$$

* We can also give other functorial interpretations of FGLs

<u>Definition</u>: Formal group = Formal group law

= d-dim'l FGL for some d.

<u>Example</u> : ⊛ $\widehat{\mathbb{G}}_m$

$$\mathbb{G}_m = \operatorname{Spec} R[t^{\pm 1}] \quad \& \quad m : \mathbb{G}_m \times \mathbb{G}_m \longrightarrow \mathbb{G}_m .$$

$$\begin{tikzcd}
xy & & t \arrow[ll, mapsto] \\
\text{e: } t=1 \quad \operatorname{Spec} R[x^{\pm 1}, y^{\pm 1}] \arrow[rr, "m"] \arrow[d, "\cong"'] & & \operatorname{Spec} R[t^{\pm 1}] \arrow[d, "\cong"] \\
\text{e: } t=0 \quad \operatorname{Spec} R[(x+1)^{\pm 1}, (y+1)^{\pm 1}] \arrow[rr, "m"] & & \operatorname{Spec} R[(t+1)^{\pm 1}] \\
(x+1)(y+1) & & t+1 \arrow[ll, mapsto]
\end{tikzcd}$$

$$(x+1)(y+1) = xy + x + y + 1$$

same as

$$x+y+xy \longleftarrow\!\shortmid \; t$$

After completion:

$$\left.\begin{array}{ccc} R[[x,y]] & \longleftarrow & R[[t]] \\ x+y+xy & \longleftarrow\!\shortmid & t \end{array}\right] \Rightarrow \widehat{\mathbb{G}}_m := x+y+xy \in R[[x,y]]$$

Functorially, if S is an R-algebra

$$\widehat{\mathbb{G}}_m(S) = \left(\mathrm{Nil}(S),\ x \cdot y = x + y + xy\right)$$

$$= \left(1 + \mathrm{Nil}(S),\ \text{usual multiplication}\right)$$

Mention the adic FF curve

(*) Similarly

$$\widehat{\mathbb{G}}_a := x + y \in R[[x, y]]$$

(*) Interesting :

If $p = 0$ in R, then

$$\mu_{p^\infty} = \varinjlim_n \mu_{p^n} = \widehat{\mathbb{G}}_m$$

(IV) Given a group-functor G of R-algs, we redefine

$$\widehat{G}(A) := \ker\left(G(A) \longrightarrow G(A_{red})\right)$$

__Note__: For a smooth group scheme G/R

$$\widehat{G_e} = \widehat{G}$$

Idea of a proof: (From Zink).

Should be able to reduce to the case when R is reduced.

Restrict to considering the functor

$$\widehat{G} : \left\{ \begin{array}{c} R\text{-algebras of the} \\ \text{form } R \oplus N \\ \text{where } N \text{ is an ideal} \\ \text{such that } N^k = 0 \end{array} \right\} \longrightarrow Ab$$

But I think it can be worked out in general.

Easier to solve here.

For an ordinary abelian variety $A/\overline{\mathbb{F}}_p$ as before, one would expect

$g = \dim(A)$

$$A[p^\infty] = \left(\mu_p^g \times \mathbb{Z}/p\mathbb{Z}^{\oplus g} \to \mu_{p^2}^g \times \mathbb{Z}/p^2\mathbb{Z}^{\oplus g} \to \cdots \right)$$

$$= \widehat{\mathbb{G}}_m^g \times (T_pA \times \mathbb{Q}_p/\mathbb{Z}_p)$$

formal torus

True more generally

A is ordinary $\Longleftrightarrow$ $\widehat{A}$ is a formal torus.