The relative Fargue-Fontaine curve

§1. Recap

Recall:

$S = \mathrm{Spa}(C, C^+)$ (for C an algebraically closed characteristic p perfectoid field.)

We have discussed, that there exists a curve

$$\begin{array}{c} X_S \\ \vdots \\ \downarrow \\ S \end{array}$$

such that

$$\left\{\begin{array}{c}\text{closed points} \\ \text{of } X_S\end{array}\right\} \cong \left\{\begin{array}{c}\text{Characteristic zero} \\ \text{untilts of } S\end{array}\right\} \Big/ \begin{array}{c}\text{Frobenius} \\ \text{equivalence}\end{array}$$

and certain period rings live in $\mathcal{O}_X$ which help studying the category

$$\mathrm{Rep}_{G_{\mathbb{Q}_p}}.$$

Can this be somehow generalised / relativized?

§2. The definition

Throughout this talk $\Bigg\}$: $S = \mathrm{Spa}(R, R^+)$ for a perfectoid pair (R, R^+)

[Ked-Liu] - assumes Tate.

[Ked-AWS17] - doesn't assume Tate.

Definition 2.1: $A_{\inf} := W(R^+)$ - complete for the (p, I)-adic topology. ($I \subseteq R^+$ is an ideal of definition)

And $\tilde{Y}_S = \mathrm{Spa}(A_{\inf}, A_{\inf})$. But $\tilde{Y}_S$ is **not** analytic. We define the analytic locus:

Theorem 2.2: The analytic locus of $\tilde{Y}_S$ is

$$\tilde{Y}_S^{an} = \left\{ v \in \mathrm{Spa}(A_{\inf}, A_{\inf}) \;\middle|\; \begin{array}{c} v([p]) \neq 0 \\ \text{or} \\ \text{there exists a topologically} \\ \text{nilpotent element } \bar{x} \in R^{+} \\ \text{such that } v([\bar{x}]) \neq 0 \end{array} \right\},$$

and it is a stably uniform adic space.

(A, A^{+}) is stably uniform if $\forall$ rational localizations $(A, A^{+}) \longrightarrow (B, B^{+})$, B is uniform, i.e. the set of power-bounded elements of B is a bounded set

Since the formation of witt-vectors is functorial, we have a Frobenius map

$$W(R^{+}) \xrightarrow{\varphi} W(R^{+})$$

<u>Definition 2.3</u> : We denote by Y_S the subspace of $\tilde{Y}_S^{an}$ consisting of all those v for which $v([P]) \neq 0$ <u>and</u> there exists a topologically nilpotent element $\bar{x} \in R^+$ for which $v([\bar{x}]) \neq 0$.

Now, it is an "not-too-hard" exercise to see that $\varphi \curvearrowright Y_S$, and infact the action is properly discontinuous. So, we define:

<u>Definition 2.4</u> : The quotiest space

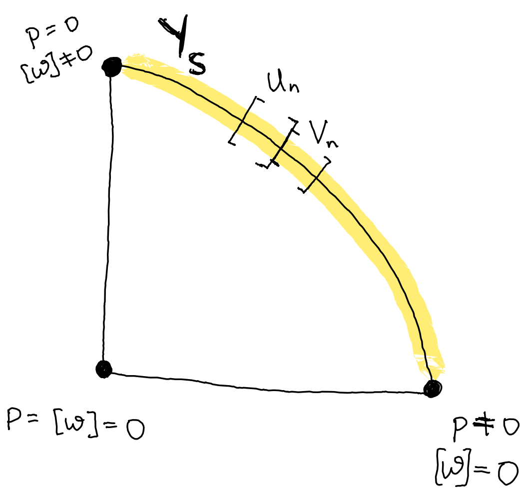
$$X_S = Y_S / \varphi^{\mathbb{Z}}$$

in the category of locally v-ringed spaces is the adic (relative) Fargue-Fontaine curve "over S", which is denoted by FF_S as well.

§3. Some drawings

Let us see some pictorial representation of FF_S

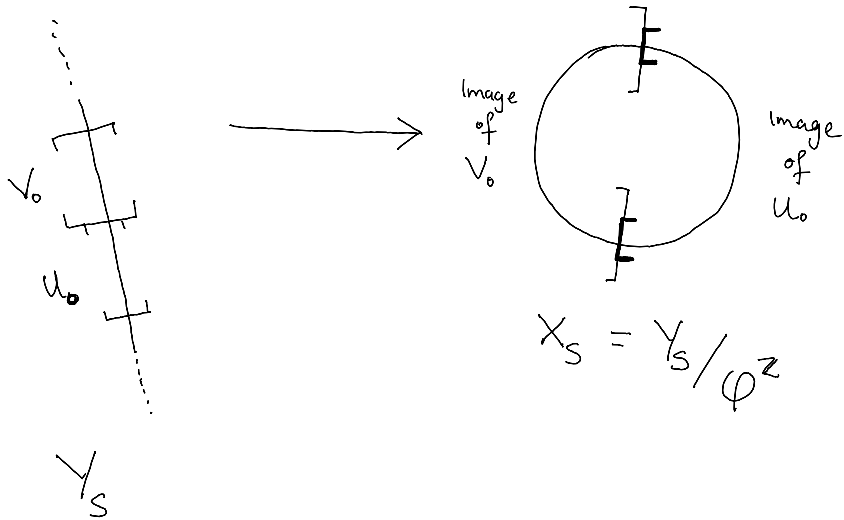
Example 3.1: The case were R is a field and $\varpi \in R^+$ a pseudo uniformizer.



$$U_n := \{ v \in Y_S \mid v(p)^{cp^n} \leq v(\varpi) \leq v(p)^{p^n} \}$$
$$V_n := \{ v \in Y_S \mid v(p)^{p^{n+1}} \leq v(\varpi) \leq v(p)^{cp^n} \}$$

for $c \in (1, p) \cap \mathbb{Q}$

φ permutes the U_n amongst themselves & similar for the V_n. Hence the action is properly discontinuous.

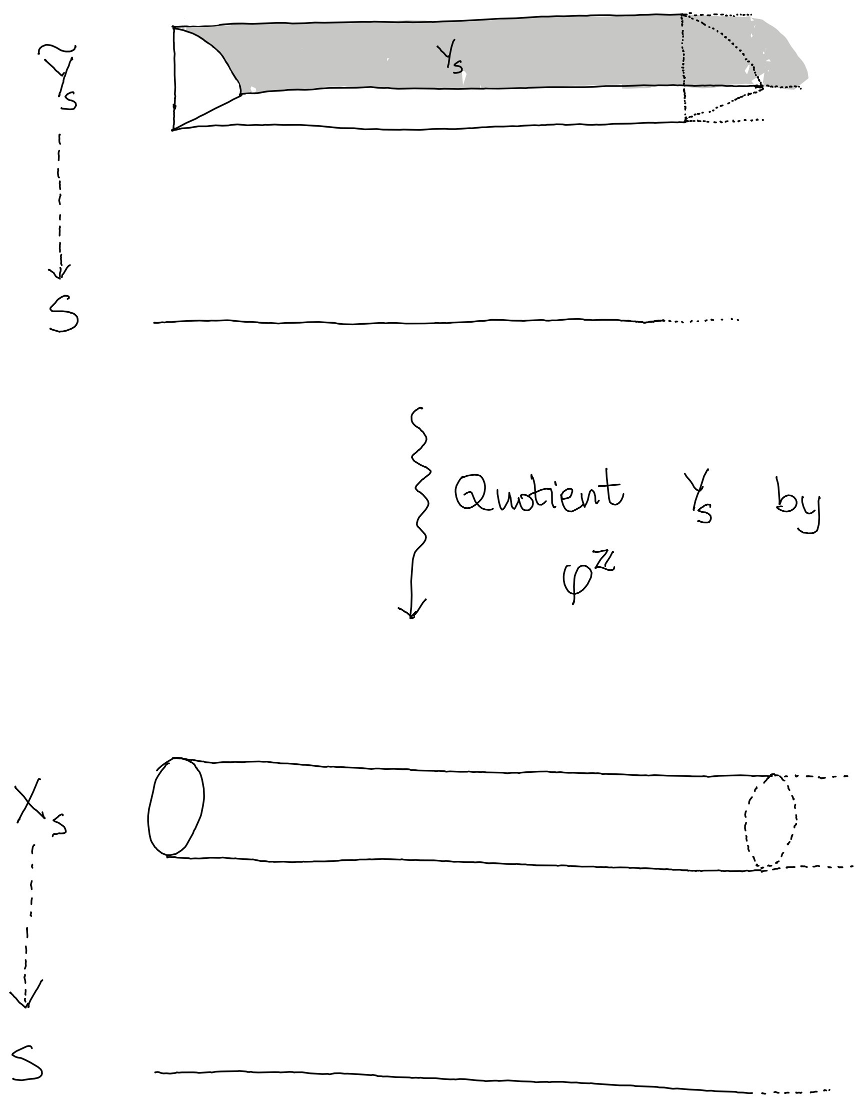


$$X_s = Y_s / \varphi^{\mathbb{Z}}$$

And,

$$\left.\begin{aligned} U_0 &\cong \text{Image of } U_0 \\ V_0 &\cong \text{Image of } V_0 \end{aligned}\right\} \text{Affinoid covering of } X_s$$

<u>Example 3.2</u> Now, for the general case.



The morphisms above are only morphisms of topological spaces, not adic spaces.

§4. Schematic (relative) FF curve

$$\left\{\begin{array}{c}\text{Vector bundles on}\\ FF_S\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\varphi\text{-equivariant}\\ \text{vector bundles on}\\ Y_S\end{array}\right\}$$

<u>Definition 4.1</u> Let $\mathcal{O}(1)$ be the line bundle on X_S corresponding to the trivial line bundle on Y_S on a generator v, with the isomorphism $\varphi^*\mathcal{O}(1) \cong \mathcal{O}(1)$ given by $1 \otimes v \longmapsto p^{-1}v$.

Under good assumptions $\mathcal{O}(1)$ can be viewed as an ample line bundle on X_S.

<u>Theorem 4.2</u>: Suppose either:

* R is Tate and $\mathcal{F}$ is a vector bundle on X_S;

or

* $(R, R^+) = (F, \mathcal{O}_F)$ for some nonarchimedean field F and $\mathcal{F}$ is a coherent sheaf on X_S.

For $n \in \mathbb{Z}$, define the twisted sheaf $\mathcal{F}(n) := \mathcal{F} \otimes \mathcal{O}(1)^{\otimes n}$. Then, for all sufficiently large n, the following statements hold:

(a) We have $H^1(X_S, \mathcal{F}(n)) = 0$.

(b) The sheaf $\mathcal{F}(n)$ is generated by finitely many global sections.

Definition 4.3 Define the graded ring

$$P_S := \bigoplus_{n=0}^{\infty} P_{S,n} \quad , \quad P_{S,n} := H^0(X_S, \mathcal{O}(n)).$$

The scheme $\mathrm{Proj}(P_S)$ is called the schematic Fargues-Fontaine curve over S.

By construction, there is a morphism

$$X_S \longrightarrow \mathrm{Proj}(P_S)$$

of locally ringed spaces.

There is also an GAGA-style theorem.

<u>Theorem 4.4</u> The morphism $X_S \to \mathrm{Proj}(P_S)$ has the following properties.

(i) Suppose that R is Tate. Then pull back of vector bundles from $\mathrm{Proj}(P_S)$ defines an equivalence of categories

(ii) Suppose $(R, R^+) = (F, \mathcal{O}_F)$ for some nonarchimedean field F. Then pullback of coherent sheaves from $\mathrm{Proj}(P_S)$ to X_S is an equivalence of categories.

(iii) In both (i) & (ii), the pullback functor preserves sheaf cohomology.

Proposition 4.5: Let $s \in H^0(X_S, \mathcal{O}(1))$ be a section which does not vanish on any fiber of X_S (i.e. its pullback to $X_{\mathrm{Spa}(F, \mathcal{O}_F)}$ does not vanish for any non-archimedean field F). Let $\mathcal{L} \subseteq \mathcal{O}$ be the image of $s \otimes \mathcal{O}(-1)$ in $\mathcal{O}$. Then the zero-locus Z of $\mathcal{L}$ is an untilt of S, & the projection $|X_S| \longrightarrow |S|$ restricts to a homeomorphism $|Z(\mathcal{L})| \cong |S|$

$$\begin{tikzcd}
S^\# \arrow[r, hook] \arrow[dr, dashed, "\text{homeomorphism}"] & FF_S \arrow[d, dashed] \\
& S
\end{tikzcd}$$

§5. Vector bundles

In Algebraic geometry, there is a famous result of Narasimhan & Sheshadri which relates

$$\begin{matrix}\text{vector bundles on} \\ \text{Riemann surfaces}\end{matrix} \longleftrightarrow \begin{matrix}\text{unitary} \\ \pi_1\text{-representations}\end{matrix}$$

In some sense analogous to this, FF:

$$\left\{\begin{matrix}\text{semistable vector bundles} \\ \text{on } FF_S \text{ of deg } 0\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{continuous rep}^n \text{ of } G_F \\ \text{on a finite dimensional} \\ \mathbb{Q}_p\text{-vector space}\end{matrix}\right\}$$

for $S = \mathrm{Spa}(F, \mathcal{O}_F)$.

Kedalaya & Liu relativised this result.

<u>Theorem 5.1</u>: For $S \in Pfd_p$, there exists an equivalence of categories

$$\left\{ \begin{array}{l} \text{étale } \mathbb{Q}_p\text{-local} \\ \text{systems on } S \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{vector bundles on} \\ FF_S \text{ which at every} \\ \text{points of } S \text{ are semistable} \\ \text{of degree zero} \end{array} \right\}$$

Moreover, this equivalence of categories equates sheaf cohomology groups on both sides.

THE END